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Abstract

Full Text

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ON A NONLINEAR PROBLEM WITH A FREE BOUNDARY

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1. Statement of the problem

Let a (sufficiently smooth) function $Q(x, y) > 0$ be given inside the unit disk $|z| \leqslant 1$, $z = x + iy$. It is required to determine a smooth curve γ, situated inside the disk $|z| < 1$, such that in the doubly connected domain G_z, bounded by γ and the unit circle $\Gamma : |z| = 1$, the following conditions are satisfied: 1°. There exists a function $\psi(x, y)$, harmonic inside G_z and continuous in $G_z + \gamma + \Gamma$. 2°. $\psi = 0$ on Γ. 3°. $\psi = c_1$, $c_1 = \text{const} \neq 0$, on γ. 4°. $|\operatorname{grad} \psi| = Q$ on γ.

Since the formulation of the problem is invariant with respect to conformal mapping, the case of an arbitrary curve Γ bounding a simply connected domain is easily reduced to the one under consideration.

Under particular assumptions on Γ and Q, this problem arises in hydrodynamics (the theory of waves in a heavy fluid, jet flows), and in these cases considerable progress has been made in the study of the problem. Thus, A. I. Nekrasov was the first to prove the existence of periodic waves in a heavy fluid (1921) (see, for example, ([4]), p. 358). M. A. Lavrent' ev ([3]) proved the existence of a solitary wave (1946). Some contemporary investigations of this range of questions are collected in the book ([5]). In the formulation considered by us, the problem was studied in the work of Beurling ([2]), where a classification of possible cases is outlined, some criteria for the existence of a solution are indicated, as well as sufficient conditions for uniqueness. In the present work an analytic method for studying the problem is applied; effectively verifiable criteria for (local) uniqueness of the problem and sufficient conditions for existence are given.

2. Reduction to an auxiliary problem

Let $\varphi(x, y)$ be the harmonic conjugate function to ψ (the potential of the flow). If one denotes by $\tilde{\nu} = \operatorname{sign} c_1 \int_\gamma Q\, ds \neq 0$ the (unknown) circulation of the flow and considers the function

$$\tau \equiv \tau(z) \equiv \exp\left\{\frac{2\pi i}{\tilde{\nu}} \chi(z)\right\}, \qquad \chi = \varphi + i\psi, \qquad \tilde{\nu} \neq 0, \tag{1}$$

then it is not difficult to prove that it realizes a conformal and one-to-one mapping of the unknown domain G_z onto the ring $G_\tau : r \leqslant |\tau| \leqslant 1$, where

$$r = \exp\{-(2\pi/\tilde{\nu})c_1\} < 1$$

(the radius r, of course, is also unknown). We therefore seek the function $z = z(\tau)$, inverse to the function (1). The last condition of the problem, in terms of the variable $z(\tau)$, takes the form

$$\left|\frac{dz}{d\tau}\right| = \frac{\lambda}{Q(x,y)} \quad \text{for } |\tau| = r, \qquad \text{where } \lambda = \frac{|\tilde{\nu}|}{2\pi} \exp\left\{\frac{2\pi|c_1|}{|\tilde{\nu}|}\right\}. \tag{2}$$

The derivative $dz/d\tau$ has no zeros in $\overline{G}_z$. Moreover, the function

$$F(\tau) = Ln\,\frac{dz(\tau)}{\lambda\, d\tau} = \ln\left|\frac{dz(\tau)}{\lambda\, d\tau}\right| + i\ \mathrm{Arg}\,\frac{dz(\tau)}{d\tau} \tag{3}$$

is single-valued, regular and analytic in G_τ. The function $z(\tau)$ is easily expressed through $F(\tau)$; with a corresponding normalization in the choice of φ we obtain the formula

$$z(\tau) = \lambda \int_1^\tau \exp\{F(t)\}\, dt + 1. \tag{4}$$

Thus, the problem has been reduced to determining the function $F(\tau)$ for which the function (4) is univalent in G_τ.

3. Reduction to integral equations. Represent the function $F(\tau)$ by the Villat formula:

$$\begin{aligned} F(\tau) = {} & \frac{i}{\pi}\int_0^{2\pi} \mu(s)\,\zeta\left(\frac{1}{i}\ln\tau - s; \pi, -i\ln r\right)\, ds \\ & - \frac{i}{\pi}\int_0^{2\pi} \mu_1(s)\,\zeta\left(\frac{1}{i}\ln\tau - s + i\ln r; \pi, -i\ln r\right)\, ds \\ & - \left(\frac{1}{2} + \frac{\eta(r)}{\pi}\ln r\right)\frac{1}{\pi}\int_0^{2\pi} \mu_1(s)\, ds + iC, \qquad \eta(r) = \frac{\pi}{12}\left[1 - 24\sum_{k=1}^{\infty}\frac{r^{2k}}{(1-r^{2k})^2}\right], \end{aligned} \tag{5}$$

where $\zeta(u;\omega,\omega')$ is the Weierstrass function with half-periods $\omega = \pi$, $\omega' = -i\ln r$; C is a real constant (see, for example, (1); we note that in this book formula (5) is given inaccurately: it lacks the preceding term on the right).

In formula (5) we have denoted $\mu(s) = \operatorname{Re} F(e^{is})$, $\mu_1(s) = \operatorname{Re} F(re^{is})$; the term iC is inessential. Comparing formulas (3) and (2), we see that $\mu_1(s) = -\ln Q(x, y)$, where $x(s) + iy(s) = z^+(re^{is})$. For the function (5) to be single-valued it is necessary and sufficient that the condition

$$A_0 \equiv \int_0^{2\pi} [\mu(s) - \mu_1(s)]\, ds = \int_0^{2\pi} [\mu(s) + \ln Q]\, ds = 0 \qquad (6)$$

be satisfied.

For brevity introduce the notation $(\zeta(u) \equiv \zeta(u; \omega, \omega'))$

$$S_0\mu(\sigma) \equiv \frac{i}{\pi} \int_0^{2\pi} \mu(s)\zeta(\sigma - s)\, ds, \qquad S\mu = \mu + S_0\mu;$$

$$S_1\mu(\sigma) = \frac{i}{\pi} \int_0^{2\pi} \mu(s) \left[\zeta(\sigma - s + i\ln r) - i\left(\frac{1}{2} + \frac{\eta}{\pi}\ln r\right)\right] ds,$$

$$S_2\mu(s) = \frac{i}{\pi} \int_0^{2\pi} \mu(s) \left[\zeta(\sigma - s - i\ln r) + i\left(\frac{1}{r} + \frac{\eta}{\pi}\ln r\right)\right] ds;$$

the first integral is understood in the sense of the Cauchy principal value. We shall regard the functions $\mu(s)$, $x(s)$, $y(s)$ and the parameter λ as unknowns. Substitute (5) into (4) and compute the limiting values $z^+(e^{i\sigma})$; requiring that the points of the circle $|\tau| = 1$ be mapped to points of the circle $|z| = 1$, we obtain the first equation

$$A_1 \equiv \left| i\lambda \int_0^{\sigma} \exp\{i\sigma + S\mu(\sigma) + S_1 \ln Q(x, y)(\sigma)\}\, d\sigma + 1 \right|^2 - 1 = 0 \qquad (7)$$

for determining $\lambda, \mu(s), x(s), y(s)$. We obtain the two other equations by computing the values $z^+(re^{i\sigma})$:

$$z^+(re^{i\sigma}) = x(\sigma) + iy(\sigma) = 1 + \lambda \int_1^r \exp\{F(t)\}\, dt +$$

$$+ i\lambda r \int_0^{\sigma} Q^{-1}(x(\sigma), y(\sigma)) \exp\{i\sigma + S_0 \ln Q(x, y)(\sigma) + S_2\mu(\sigma)\}\, d\sigma. \qquad (8)$$

Thus, the problem under consideration is equivalent to the system of equations (7), (8) for the unknowns $\lambda, \mu(s), x(s), y(s)$, and we must find its solution ensuring the single-valuedness of the functions (4), (5).

The results given below pertain to the particular case in which $Q(x, y) \equiv q(\rho)$, where $\rho^2 = x^2 + y^2$. In this case the unknowns will be $\lambda, \mu(s)$, and $\rho(s)$. To determine them we obtain equation (7) together with the equation

$$A_2 \equiv \rho^2(\sigma) - \left|1 + \lambda L + i\lambda r \int_0^\sigma \frac{1}{q[\,\rho(\sigma)\,]} \exp\{i\sigma + S_2\mu + S_0 \ln q(\rho)(\sigma)\}\, d\sigma\right|^2 = 0 \tag{8'}$$

(to obtain this equation, one must take the moduli of both sides of equality (8) and square them; L is the functional corresponding to the first integral in (8)).

4. Spaces and operators. Denote by E the set of triples $\omega = (\lambda, \mu, \rho)$ and suppose that $\mu, \rho \in \operatorname{Lip}\beta$, $\beta > 0$, with $\mu(s)$ 2π-periodic. Introducing on E the norm

$$\|\omega\|_E = |\lambda| + \|\mu\|_{\operatorname{Lip}\beta} + \|\rho\|_{\operatorname{Lip}\beta},$$

we obtain a complete normed linear Banach space. The radius r and the parameter λ are related by the equation $\lambda r \ln r = -|c_1|$, whence, in a neighborhood of any $r \neq e^{-1}$, a single-valued branch $r = r(\lambda)$ is determined. Continuing $r(\lambda)$ and $q(\rho)$ to all real λ and ρ with preservation of sufficient smoothness, we extend the operators A_0, A_1, A_2 to all of E. Denote by E_1 the set of triples $\omega_1 = (\mu_0, \mu_1, \mu_2)$, where μ_0 is a number, μ_1, μ_2 are functions of s, with $\mu_2 \in \operatorname{Lip}\beta$, $\mu_1 \in \operatorname{Lip}\beta$, $\mu_1' \in \operatorname{Lip}\beta$, $\mu_1(0) = 0$. Equipped with the norm

$$\|\omega_1\|_{E_1} = |\mu_0| + \|\mu_1\|_{\operatorname{Lip}\beta} + \|\mu_1'\|_{\operatorname{Lip}\beta} + \|\mu_2\|_{\operatorname{Lip}\beta},$$

the set E_1 becomes a Banach space. It is proved that the three operators A_0, A_1, A_2 define a continuous and continuously differentiable mapping $\omega_1 = \varphi(\omega)E$ into E_1.

In attempting to find symmetric domains $G_z : \ |z| = \rho_0$, we obtain

$$\psi = c_1 \ln \rho / \ln \rho_0,$$

where ρ_0 is determined from the equation

$$\rho_0 q(\rho_0) \ln \rho_0 = -|c_1|. \tag{9}$$

If ρ_0 is a root of this equation, then the triple

$$\omega_0 = (\lambda_0, \mu_0, \rho_0) = (q(\rho_0); \ln q(\rho_0); \rho_0),$$

as may be verified, satisfies the equation $\varphi(\omega_0) = 0$. The problem consists in finding a solution of the equation $\varphi(\omega) = 0$ close to ω_0.

5. Linearized system. Uniqueness theorem. Denote $X = (\Delta\lambda, h, l)$, $X_1 = (\mu_0, \mu_1, \mu_2)$ and consider the inhomogeneous equation

$$\varphi'(\omega_0; X) = X_1, \qquad X \in E, \qquad X_1 \in E_1, \tag{10}$$

where $\varphi'(\omega_0; X)$ is the Fréchet derivative of the mapping $\omega_1 = \varphi(\omega)$ at the point ω_0. Equation (10) is equivalent to a certain linear boundary-value problem in an annulus for an analytic function, and by expansion in a Laurent series this problem is investigated completely.

Theorem 1. *Suppose that $q(\rho)$ has a second derivative continuous in the Hölder sense and is positive in a neighborhood of the root ρ_0 of equation (9). Suppose, moreover, that $q'(\rho_0) \neq 0$, $\rho_0 \neq e^{-1}$, and that the conditions*

$$\frac{q(\rho_0)}{q'(\rho_0)}\rho_0^n + \frac{n-1}{n+1}\frac{q(\rho_0)}{q'(\rho_0)}\rho_0^{-n} - \frac{\rho_0^{1-n}}{1+n} + \frac{\rho_0^{1+n}}{1+n} \neq 0, \qquad n = 1, 2, \dots,$$

$$\frac{q(\rho_0)}{q'(\rho_0)} + \rho_0 - \frac{\rho_0}{1+\ln\rho_0} \neq 0. \tag{11}$$

In order for equation (10) to be solvable, it is necessary and sufficient that the right-hand side X_1 satisfy the condition

$$\eta(\rho_0)\mu_0 - H(\mu_1, \mu_2) = 0, \tag{12}$$

where $H(\mu_1, \mu_2)$ is a certain (explicitly writable) linear functional.

Under the conditions (11), the homogeneous equation ($X_1 = 0$) has only the trivial solution.

The cases when $\rho_0 = e^{-1}$ or $q'(\rho_0) = 0$ have also been considered. If at least one of the conditions (11) is violated, then the homogeneous equation has nontrivial solutions, which are written down explicitly.

Theorem 2 (on local uniqueness). *Under the conditions of Theorem 1, the original problem has no nontrivial (i.e., different from circular) solutions in some (generally speaking, small) neighborhood of the function $\rho(s) = \rho_0$ (in the sense of the metric $\operatorname{Lip}\beta$, $\beta > 0$).*

The proof is based on the fact that, under the conditions of Theorem 1, the Fréchet derivative $\varphi'(\omega_0; X)$ has a left inverse operator.

6. Existence theorem. The problem considered by us includes the case of periodic waves in a heavy fluid. The corresponding function $q(\rho)$ has the form

$$q(\rho) = (2\pi\rho)^{-1}\sqrt{C + \pi^{-1} gl \ln\rho},$$

where l is the wavelength and C is Bernoulli' s constant. Therefore we shall assume that q, in addition to the variable ρ, contains a certain number of parameters $\nu = (\nu_1, \nu_2, \dots, \nu_n)$. The mapping $\omega_1 = \varphi(\omega; \nu)$ will also depend on

ν. By ρ_0 we shall mean the root of equation (9) for some fixed values $\nu = \nu^0$. Under the conditions of Theorem 1 it is proved that, if $\eta(\rho_0) \neq 0$, there exists an n-parameter solution $\lambda = \lambda(\nu)$, $\mu = \mu(\nu, s)$, $\rho = \rho(\nu, s)$ of the system of equations $A_1 = 0$, $A_2 = 0$, defined in a neighborhood of (ρ_0, ν^0) and such that $\lambda(\nu^0) = \lambda_0$, $\mu(\nu^0, s) = \mu_0$, $\rho(\nu^0, s) = \rho_0$.

In order that the function (4) be single-valued, it is obviously necessary that

$$f_1(\nu_1, \nu_2, \dots, \nu_n) = \operatorname{Im} \int_{|t|=1} \exp\{F(t,\nu)\}\, dt = 0, \tag{13}$$

where $F(t,\nu)$ is the function constructed by a formula of the form (5) for the given solution. We note that $f_1(\nu_1^0, \nu_2^0, \dots, \nu_n^0) = 0$, for in this case the function (1) coincides with z. Direct calculations show that

$$\left.\frac{\partial f_1}{\partial \nu_1}\right|_{\nu=\nu^0} = \frac{[4\eta(\rho_0) q_{\nu_1}(\rho_0, \nu^0)}{\rho_0 q^2(\rho_0, \nu^0)} \frac{1-\rho_0}{\rho_0 q(\rho_0, \nu^0)/q_\rho(\rho_0 \nu^0) + \frac{1}{2}(\rho_0^2 - 1)} \times \\ \times \left[\frac{1}{1-\rho_0^2} + \frac{2\eta(\rho_0)}{\pi}(\rho_0 - 1)\right]. \tag{14}$$

Hence it follows that if

$$q'_\rho(\rho_0, \nu^0) \neq 0, \qquad q'_{\nu_1}(\rho_0, \nu^0) \neq 0, \qquad \eta(\rho_0) \neq 0, \tag{15}$$

then the partial derivative of the function (13) with respect to ν_1 at the initial point $\nu = \nu^0$ is nonzero. Consequently, there exists a function $\nu_1 = \nu_1(\nu_2, \dots, \nu_n)$ on which the functional (13) vanishes identically. It is also proved that on this function the functional $f_2(\nu) \equiv A_0(\mu(s,\nu), \rho(s,\nu))$ also vanishes, and that the function (4) is single-valued.

Theorem 3 (existence theorem). *Let the function q depend on ρ and on n (essential) parameters $\nu_1, \nu_2, \dots, \nu_n$, and suppose that at the point (ρ_0, ν^0) the conditions of Theorem 1 are satisfied. If q is continuously differentiable with respect to the parameters ν and the conditions (15) are satisfied, then, in some neighborhood of $\rho = \rho_0$, the original problem has an $(n-1)$-parameter family of solutions different from the trivial solution $\rho = \rho_0$.*

We note that the variable c_1 from the third condition can also be regarded as a parameter of the problem. In this case there exists an n-parameter family of solutions. As a corollary we obtain from this an existence theorem for a two-parameter family of periodic waves.

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Note: Figure translations are in progress. See original paper for figures.