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Abstract

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MATHEMATICAL PHYSICS

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EXISTENCE OF A PHASE TRANSITION IN THE TWO-DIMENSIONAL AND THREE-DIMENSIONAL ISING MODELS

(Presented by Academician A. N. Kolmogorov on 18 IX 1964)

Let $\mathfrak{V}_l$ be a system of $V_l = l^\nu$ points $X = (x_1, \dots, x_\nu)$, $x_i = 1, \dots, l$, forming a cube with side l in the ν-dimensional integer lattice (the physically interesting cases are $\nu = 1, 2, 3$). By a configuration of N particles we shall mean a set b of N points of $\mathfrak{V}_l$. We denote the set of such configurations by $\mathfrak{B}(N, l)$. In the Ising model, to each configuration $b = (X_1, \dots, X_N)$ there corresponds the potential

$$U(b) = -\frac{1}{2}\sum_{i \neq j} U(X_i, X_j), \tag{1}$$

where $U(X_i, X_j) = 1$ if the distance from X_i to X_j is equal to 1, and is equal to zero in the other cases. The Gibbs distribution is the probability distribution in which the probability of an event $B \subset \mathfrak{B}(N, l)$ is defined as

$$\mathfrak{P}_{N,l}(B) = \frac{1}{Z(N, l)} \sum_{b \in B} \exp\left\{-\frac{U(b)}{T}\right\}, \tag{2}$$

where the normalizing factor is

$$Z(N, l) = \sum_{b \in \mathfrak{B}(N,l)} \exp\left\{-\frac{U(b)}{T}\right\} \tag{3}$$

and $T > 0$ is the temperature of the system. It is known that if the sequence N_l is such that $V_l/N_l \to v$ $(l \to \infty)$, $1 < v < \infty$, then there exists the limit

$$\lim_{l \to \infty} \frac{1}{N_l} \log Z(N_l, l) = f(v, T). \tag{4}$$

The point (v_0, T_0) is called a phase-transition point if $f(v, T_0)$ is a linear function of v on some interval containing v_0. (For the concepts introduced here, see ([1,2]);

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for the results of previous investigations relating to the one-dimensional and two-dimensional models, see ([1,3]).)

The purpose of this note is to indicate the main steps of a mathematically rigorous* proof of the following theorem:

Theorem. Suppose, for $\nu > 1$,

$$Q(T) = \frac{eS_\nu \left[1 - \exp\left(-\frac{1}{2T}\right)\right]}{3\left[1 + \exp\left(-\frac{\nu}{T}\right)\right]} \sum_{k=2\nu}^{\infty} \left[3\exp\left(-\frac{1}{2T}\right)\right]^k k^{\nu/\nu-1}, \tag{5}$$

where S_ν is the volume of the ν-dimensional sphere with unit surface area

$$\left(S_2 = \frac{1}{4\pi}, \quad S_3 = \frac{1}{6\sqrt{\pi}}\right).$$

Then, if

* In view of the fact that in contemporary physical literature the term "mathematically rigorous" is applied, for convincingness, to arguments of a rather varied logical nature, we note that here it is used in the sense generally accepted in the mathematical literature.

$$Q(T) < \frac{1}{v} < 1 - Q(T), \tag{6}$$

then a phase transition takes place at the point (v, T).

Unlike the analytic methods described in the cited works, which make it possible (where they exist) to describe completely the boundaries of the phase-transition region, the qualitative method developed here gives only one-sided estimates of these boundaries, and rather crude ones at that. However, this new method is equally applicable to two-dimensional and three-dimensional problems and admits generalizations to more complicated potentials.

The proof is based on the following lemmas.

For any subcube H, embedded in the fundamental cube, with sides parallel to the sides of the fundamental cube, denote by $\mu_H(b)$, $b \in \mathfrak{B}(N, l)$, the number of points $X_i \in H$.

Lemma 1. *In order that a phase transition occur at the point (v, T), it is necessary and sufficient that, for some $\varepsilon > 0$, for any sequence of subcubes H_l with side lengths k_l such that*

$$0 < \lim_{l\to\infty} \frac{k_l}{l} < \overline{\lim}_{l\to\infty} \frac{k_l}{l} < 1 \tag{7}$$

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and any sequence N_l such that $V_l/N_l \to v$,

$$\lim_{l\to\infty} \frac{1}{N_l} \log \mathfrak{P}_{N_l,l} \left\{ \left| \frac{\mu_{H_l}}{k_l'} - \frac{1}{v} \right| \geq \varepsilon \right\} = 0. \tag{8}$$

The proof of this lemma is a refinement of the well-known argument connected with deriving the large canonical ensemble from the small one.

Corollary. *If, under the conditions of Lemma 1, the probability*

$$\mathfrak{P}_{N_l,l} \left\{ \left| \frac{\mu_{H_l}}{k_l^\nu} - \frac{1}{v} \right| \geq \varepsilon \right\}$$

does not tend to zero for some $\varepsilon > 0$, then a phase transition takes place at the point (v, T).

The proof of the theorem is based on verifying that, under condition (5), the hypotheses of the formulated corollary are satisfied.

We shall carry out the subsequent constructions, for simplicity, for $\nu = 2$, although the generalizations to higher dimensions are straightforward. With each point of the lattice associate a square of side 1 centered at this point. By $G(b)$, $b \in \mathfrak{B}(N, l)$, we shall denote the region which is the union of the squares centered at the points of b. The boundary of the region $G(b)$ naturally decomposes into closed, nonintersecting polygonal lines, which we shall call cycles. Number the cycles in decreasing order of their lengths and denote these lengths by $\Gamma_1(b), \dots, \Gamma_{K(b)}(b)$, where $K(b)$ is the number of cycles.

Lemma 2. *Let $M < \infty$, $D < \infty$, $\delta > 0$ be fixed numbers. Then, for all sufficiently small α, for sufficiently large l, and for any arrangement b such that the conditions*

$$\sum_{i=1}^{M} \Gamma_i(b) \leq Dl, \tag{9}$$

$$\sum_{i=M+1}^{K(b)} (\Gamma_i(b))^2 \leq 4\pi \min(N_l, V_l - N_l)(1 - \delta), \tag{10}$$

are satisfied, there exists a subsquare $H(b)$ of the fundamental square with side $[\alpha l]$ and lower-left point with coordinates of the form $(r_1[\alpha l] + 1,\ r_2[\alpha l] + 1)$, where r_1, r_2 are integers, such that for

$$\varepsilon < \frac{\delta}{10} \min(v^{-1}, 1 - v^{-1})$$

the quantity

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$$\left|\frac{\mu_{H(b)}(b)}{[\alpha l]^2} - \frac{1}{v}\right| \geq \varepsilon. \tag{11}$$

The proof of this lemma is based on the fact that, by a well-known theorem of geometry, the area inside a curve of length $\Gamma_i(b)$ does not exceed $4\pi(\Gamma_i(b))^2$, and therefore (10) shows that inside the cycles $\Gamma_{M+1}(b), \dots, \Gamma_{K(b)}(b)$ there can lie neither $G(b)$ nor its complement. On the other hand, if one of the "large" cycles ($i = 1, \dots, M$) does not contain within itself other "large" cycles, then inside it there belong the interiors of the "small" cycles, or all points of $G(b)$, or all points of the complement of $G(b)$.

Lemma 3. *If condition (6) of the theorem is satisfied, then there exist $M < \infty$, $D < \infty$, $\delta > 0$ such that the probability that conditions (9) and (10) are satisfied tends to 1 as $l \to \infty$.*

From Lemmas 2, 3 and the corollary to Lemma 1, the assertion of the theorem follows easily. To prove Lemma 3 we use an auxiliary ensemble. Let $\overline{\mathcal{B}}(l)$ be the sum of $\mathcal{B}(N,l)$ over all $N = 0, 1, \dots, V_l$. Put, for $B \subset \overline{\mathcal{B}}(l)$,

$$\overline{\mathcal{P}}_l(B) = \frac{1}{\overline{Z}(l)} \sum_{b \in B} \exp\left\{-\frac{\Gamma(b)}{2T}\right\}, \tag{12}$$

where

$$\overline{Z}(l) = \sum_{b \in \overline{\mathcal{B}}(l)} \exp\left\{-\frac{\Gamma(b)}{2T}\right\} \tag{13}$$

and $\Gamma(b)$ is the total length of the boundary of the region $G(b)$.

Lemma 3 follows easily from the following Lemmas 4, 5 and 6.

Lemma 4. *For any $\gamma > 0$ and all N such that $V_{lQ}(T)(1+\gamma) > N > V_l - V_{lQ}(T)(1+\gamma)$, uniformly in such N,*

$$\lim_{l \to \infty} \frac{Z(N,l)\exp\{l^2/T\}}{\overline{Z}(l)} = \infty. \tag{14}$$

Lemma 5. *For any $\varepsilon > 0$*

$$\lim_{l \to \infty} \overline{\mathcal{P}}_l \left\{ \sum_{i=1}^{K(b)} (\Gamma_i(b))^2 \geq 4\pi Q(T)(1+\varepsilon) \right\} = 0. \tag{15}$$

Lemma 6. *Let $S(M, D, \varepsilon)$ be the set of arrangements $b \in \overline{\mathcal{B}}(l)$ such that*

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$$\sum_{i=1}^{M}\Gamma_i(b) \le Dl, \qquad \sum_{i=M+1}^{K(b)}(\Gamma_i(b))^2 \le 4\pi Q(T)(1+\varepsilon). \tag{16}$$

Then for any $\varepsilon > 0$ one can choose numbers M and D so that

$$\lim_{l\to\infty}\overline{\mathcal{P}}_l(b \in S(M,D,\varepsilon))\exp\left\{\frac{l^2}{T}\right\} = 0. \tag{17}$$

Lemmas 5 and 6 follow, in turn, from the estimate

$$\overline{\mathcal{P}}_l\{\Gamma_r(b) \ge k\} \le \left(\frac{V_l e(1-\exp\{-(2T)^{-1}\})}{3r(1+\exp\{-2T^{-1}\})(1-3\exp\{-(2T)^{-1}\})}[3\exp\{-(2T)^{-1}\}]^k\right)^r. \tag{18}$$

Estimate (18) follows from the following geometric lemma:

Lemma 7. *The total number of geometrically distinct cycles passing through a fixed point and having length not exceeding k is no more than 3^{k-1}.*

A presentation with complete proofs is being published in the journal *Theory of Probability and Its Applications.*

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Note: Figure translations are in progress. See original paper for figures.

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