THE CAUCHY PROBLEM FOR THE LINEARIZED BOLTZMANN EQUATION

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Abstract

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A. A. ARSEN' EV

THE CAUCHY PROBLEM FOR THE LINEARIZED BOLTZMANN EQUATION

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1. Recently many works have appeared devoted to the Cauchy problem for the linearized Boltzmann equation in the kinetic theory of gases ([1,2]). Interest in this topic is due to the desire to obtain, proceeding from molecular-kinetic concepts, equations of motion for rarefied gases that are as accurate as possible. Of course, the description of the state of a gas by means of the linearized Boltzmann equation can be regarded only as a rough approximation, valid, perhaps, only in the case of extremely small deviations from the state of equilibrium; nevertheless, even for this simplified problem there is neither a clear mathematical formulation nor a solution, and the results available on this subject are very vague.

2. The problem is as follows. Suppose that at the initial instant $t = 0$ the distribution function of the number of particles in phase space $F(\mathbf{x}, \mathbf{v}, t)$ can be written in the form

$$F(\mathbf{x}, \mathbf{v}, 0) = \left(\frac{1}{\pi}\right)^{3/2} e^{-v^2} + \mu e^{-1/2v^2} f(\mathbf{x}, \mathbf{v}, 0),$$

where $|\mu| \ll 1$ and $f(\mathbf{x}, \mathbf{v}, 0)$ is a sufficiently good function. Then, to within terms of order μ^2, the function $f(\mathbf{x}, \mathbf{v}, t)$ will satisfy the equation:

$$\frac{\partial f}{\partial t} + \mathbf{v}\frac{\partial f}{\partial \mathbf{x}} = \frac{1}{\varepsilon} L f,$$

$$\mathbf{x} = (x_1, x_2, x_3), \qquad \mathbf{v} = (v_1, v_2, v_3), \qquad 0 \le |\mathbf{x}| \le \infty; \quad 0 \le |\mathbf{v}| \le \infty,$$

$$f\big|_{t=0} = f(\mathbf{x}, \mathbf{v}, 0), \tag{1}$$

where L is a certain linear operator, independent of both $\mathbf{x}$ and t. Equation (1) is called the **linearized Boltzmann equation**.

3. We shall state without proof the principal results of our work.

Definition 1. By a **solution of equation (1)** we mean a function $f(\mathbf{x}, \mathbf{v}, t)$ which, for every $t > 0$, belongs to $L_2(\mathbf{x} \otimes \mathbf{v})$, and whose Fourier transform

$$\hat{f}(\mathbf{k}, \mathbf{v}, t) = \int e^{i\mathbf{k}\mathbf{x}} f(\mathbf{x}, \mathbf{v}, t)\, d\mathbf{x}$$

for almost all $\mathbf{k}$ is a solution (understood in the sense of the definition of R. Phillips ([3]), p. 631) of the following abstract Cauchy problem:

$$\frac{\partial \hat{f}}{\partial t} = -i(\mathbf{v}; \mathbf{k})\hat{f} + \frac{1}{\varepsilon} L\hat{f} = A(\mathbf{k})\hat{f}, \qquad \hat{f}\big|_{t=0} = \hat{f}_0. \tag{2}$$

In equation (2) the function $\hat{f}(\mathbf{k}, \mathbf{v}, t)$ is regarded as an element of $L_2(\mathbf{v})$, depending on $\mathbf{k}$ and t. The scalar product and the norm will be understood precisely in the sense of this space.

Assume that the operator L has the following properties:

I. $Lf = -\nu(\mathbf{v})f + Gf$, where $\nu(\mathbf{v})$ is integrable with its square on every bounded set and depends only on $|\mathbf{v}|$, while G is completely continuous–

integral operator with symmetric kernel $G(\mathbf{v}_1, \mathbf{v}_2)$, where $G(\mathbf{v}_1, \mathbf{v}_2)$ depends only on $|\mathbf{v}_1|$, $|\mathbf{v}_2|$, and $(\mathbf{v}_1, \mathbf{v}_2)$.

II. $\langle f, Lf\rangle = \langle Lf, f\rangle \leqslant 0$; $\langle f, Lf\rangle = 0$ only in the case when $Lf = 0$, and $Lf = 0$ only in the case when f is a linear combination of the following 5 functions:

$$u_1 = \frac{\sqrt{2}}{(\pi)^{3/4}}\, v_1 e^{-1/2v^2}, \qquad u_2 = \frac{\sqrt{2}}{(\pi)^{3/4}}\, v_2 e^{-1/2v^2}, \qquad u_3 = \frac{\sqrt{2}}{(\pi)^{3/4}}\, v_3 e^{-1/2v^2},$$

$$u_4 = \left(\frac{1}{\pi}\right)^{3/4} e^{-1/2v^2}, \qquad u_5 = \sqrt{\frac{2}{3}}\left(\frac{1}{\pi}\right)^{3/4} (v^2 - 3/2)e^{-1/2v^2}.$$

III. There exist positive constants $a > 0$ and $b > 0$ such that for all $\mathbf{v}$

$$\nu(\mathbf{v}) > a + b|\mathbf{v}|.$$

IV.

$$\int |G(\mathbf{v}_1, \mathbf{v}_2)|\, dv_2 < c < \infty,$$

where c is independent of $\mathbf{v}_1$.

Suppose that the initial function $f_0(\mathbf{x}, \mathbf{v})$ is such that:

V.

$$\hat{f}_0(\mathbf{k}, \mathbf{v}) \in D(A(\mathbf{k})).$$

VI.
$$\int (1+|\mathbf{k}|^N)\,|\hat{f}_0(\mathbf{k})|\,d\mathbf{k} < \infty, \qquad 1 \leqslant N \leqslant N_0 < \infty.$$

Under these assumptions we prove that a solution of equation (1) exists, and moreover:

1)
$$\int |f(\mathbf{x},\mathbf{v},t)|^2\,dx\,dv \to 0, \qquad t \to \infty.$$

2) Weak
$$\lim_{t\to\infty} t^{3/2} f(\mathbf{x},\mathbf{v},t) =$$

$$= \frac{1}{4\pi^3}\left[\frac{1}{5}\left(\frac{1}{\lambda_1}\right)^{3/2} e^{-1/2v^2}(v^2-2.5)\int e^{-1/2v^2}(v^2-2.5) f_0(\mathbf{x},\mathbf{v})\,dx\,dv + \right.$$

$$\left. + \frac{1}{3}\left(\frac{1}{\lambda_2}\right)^{3/2} \sum_{i=1}^{3} e^{-1/2v^2} v_i \int e^{-1/2v^2} v_i f_0(\mathbf{x},\mathbf{v})\,dx\,dv\right],$$

where λ_1 and λ_2 are certain positive constants.

4. Of great interest from the physical point of view is the asymptotic behavior, as $\varepsilon \to 0$, of the solution of equation (1) (ε is the ratio of the mean free path to the scale of the phenomena under study). In this direction the following results have been obtained (it is assumed that conditions I–VI are satisfied):

Theorem 1.

Strong
$$\lim_{\varepsilon\to 0} f(\mathbf{x},\mathbf{v},t;\varepsilon) = \tilde{f}(\mathbf{x},\mathbf{v},t) := f_1(\mathbf{x},\mathbf{v}) + f_2(\mathbf{x},\mathbf{v}) + f_3(\mathbf{x},\mathbf{v},t),$$

where

$$f_1 = \frac{2}{5}\left(\frac{1}{\pi}\right)^{3/2} e^{-1/2v^2}(v^2-2.5)\int e^{-1/2v^2}(v^2-2.5) f_0(\mathbf{x},\mathbf{v})\,dv,$$

$$f_2 = \sum_{i=1}^{3}\langle u_i, f_0\rangle u_i + \sum_{i,j=1}^{3}\langle u_i, \varphi_{ij}\rangle u_j; \qquad \varphi_{ij} = \frac{1}{4\pi}\int \frac{1}{r_{xx'}}\frac{\partial^2 f_0}{\partial x'_i \partial x'_j}\,dx';$$

$f_3(\mathbf{x},\mathbf{v};t)$ is the solution of the following Cauchy problem for the wave equation:

$$\partial^2 f_3/\partial t^2 = \lambda_0^2 \Delta f_3, \qquad \lambda_0 = \sqrt{5/6},$$

$$f_3(\mathbf{x}, \mathbf{v}; 0) = \frac{2}{5}\langle u_4, f_0\rangle u_4 + \frac{2}{5}\langle u_5, f_0\rangle u_5 + \frac{\sqrt{6}}{5}\langle u_4, f_0\rangle u_5 +$$

$$+\frac{\sqrt{6}}{5}\langle u_5, f_0\rangle u_4 - \sum_{i,j=1}^{3}\langle u_i, \varphi_{ij}\rangle u_j,$$

$$\left.\frac{\partial f_3}{\partial t}\right|_{t=0} = -\sum_{j=1}^{3}\left[\frac{1}{\sqrt{2}}\left\langle u_4, \frac{\partial f_0}{\partial x_j}\right\rangle u_j + \frac{1}{\sqrt{3}}\left\langle u_5, \frac{\partial f_0}{\partial x_j}\right\rangle u_j + \right.$$

$$\left. + \frac{1}{\sqrt{2}}\left\langle u_j, \frac{\partial f_0}{\partial x_j}\right\rangle u_4 + \frac{1}{\sqrt{3}}\left\langle u_j, \frac{\partial f_0}{\partial x_j}\right\rangle u_5\right].$$

It can be shown that, in dimensional units, λ_0 coincides with the adiabatic speed of sound in an ideal monatomic gas, while the moments of the function $\tilde{f}(\mathbf{x}, \mathbf{v}; t)$ satisfy the usual equations of acoustics.

Theorem 2. The following estimate holds

$$\|f(\mathbf{x}, \mathbf{v}, t; \varepsilon) - W(\mathbf{x}, \mathbf{v}, t; \varepsilon)\| \le C\left[\frac{\varepsilon^2 t}{1+(\varepsilon t)^3} + \frac{\varepsilon}{1+(\varepsilon t)^2}\right] + O(e^{-at/\varepsilon}), \quad a > 0, \tag{3}$$

where

$$W(\mathbf{x}, \mathbf{v}, t; \varepsilon) = W_1 + W_2 + W_3,$$

$$W_1 = \left(\frac{1}{4\pi\lambda_1\varepsilon t}\right)^{3/2}\int \exp\left[-(\mathbf{x}-\mathbf{x}')^2/4\lambda_1\varepsilon t\right] f_1(\mathbf{x}', \mathbf{v})\, d\mathbf{x}',$$

$$W_2 = \left(\frac{1}{4\pi\lambda_2\varepsilon t}\right)^{3/2}\int \exp\left[-(\mathbf{x}-\mathbf{x}')^2/4\lambda_2\varepsilon t\right] f_2(\mathbf{x}', \mathbf{v})\, d\mathbf{x}',$$

$$W_3 = \left(\frac{1}{4\pi\lambda_3\varepsilon t}\right)^{3/2}\int \exp\left[-(\mathbf{x}-\mathbf{x}')^2/4\lambda_3\varepsilon t\right] f_3(\mathbf{x}', \mathbf{v}, t)\, d\mathbf{x}'.$$

We have developed a method for obtaining estimates of type (3) with any desired degree of accuracy in ε; however, the scope of this note does not allow us to present the corresponding results here.

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Moscow State University
named after M. V. Lomonosov

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[1] H. Grad, *Phys. Fluids*, **6**, No. 2 (1963). [2] L. Sirovich, *Phys. Fluids*, **6**, No. 1, 2, 10 (1963). [3] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.