MATHEMATICS

A. A. LORENTS

1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. A. LORENTS

COEFFICIENT-FREE EQUATIONS IN FREE GROUPS

(Presented by Academician P. S. Novikov, 17 VII 1964)

In the present note we consider properties of "coefficient-free equations" in free groups, the precise definition of which is given below. Theorem 1 establishes for free groups essentially the same result as was obtained for equations in words by the Bulgarian mathematicians D. Skordev and Bl. Sendov ([1]). In addition, the note considers systems of coefficient-free equations and gives a method for effectively describing the set of solutions for certain systems of a special form. Along with new notions we shall use the terms introduced by the author in ([2]).

Let a free group X with generators $X_1, X_2, \dots, X_n$ be given. We shall call expressions of the form

$$W(X_1, X_2, \dots, X_n) = 1 \tag{1}$$

coefficient-free equations in the unknowns $X_1, X_2, \dots, X_n$, if $W(X_1, X_2, \dots, X_n)$ is an element of the group X such that for every i $(i = 1, 2, \dots, n)$ either X_i or X_i^{-1} occurs in $W(X_1, X_2, \dots, X_n)$.

We shall call equation (1) **normal** if the element $W(X_1, X_2, \dots, X_n)$ is cyclically reduced.

Let a free group $\mathfrak{G}$ of rank r and a sequence of elements of this group $S = \{S_1, S_2, \dots, S_n\}$ be given. Denote by $W(S_1, S_2, \dots, S_n)$ the element of the free group $\mathfrak{G}$ obtained by substituting the terms S_i of the sequence S in place of X_i in $W(X_1, X_2, \dots, X_n)$.

It is natural to call a **solution** of equation (1) in the free group $\mathfrak{G}$ any sequence S of elements of the free group $\mathfrak{G}$ such that $W(S_1, S_2, \dots, S_n) = 1$*.

Definition 1. Let sequences $S = \{S_1, S_2, \dots, S_n\}$, $T = \{T_1, T_2, \dots, T_n\}$ of elements of a free group $\mathfrak{G}$ be given. We define inductively the relation of **association** between S and T:

1. S associates T if $T_j = S_j^e S_k^\varepsilon$ $(e = \pm 1,\ \varepsilon = \pm 1,\ 1 \le j, k \le n,\ j \ne k)$ and for all i $(i = 1, 2, \dots, n)$, where $i \ne j$, $T_i = S_i$.

2. If the sequence S associates the sequence S' and S' associates the sequence T, then S associates T.

Definition 2. Suppose that free groups $\mathfrak{G}_1$ of rank r_1 and $\mathfrak{G}_2$ of rank r_2 ($r_2 < r_1$) are given. Fix the sequence

$$S = \{S_1, S_2, \dots, S_{r_2}\} \tag{2}$$

of elements of the group $\mathfrak{G}$.

By the symbol Φ_S we shall denote the mapping of the group $\mathfrak{G}_2$ into $\mathfrak{G}_1$ produced according to the following rules:

1. The image in $\mathfrak{G}_1$ of a generator G_{2i} ($i = 1, 2, \dots, r_2$) of the group $\mathfrak{G}_2$ is the term S_i from S.
2. Let $U = VG_{2i}^e$ ($e = \pm 1$) be an element of the group $\mathfrak{G}_2$, and suppose that the image of the element V in $\mathfrak{G}_1$ is the element V'; then the image of U in $\mathfrak{G}_1$ is the element $U = V'S_i^e$.

---

* The identity element of the group $\mathfrak{G}$ is denoted by 1.

The mapping Φ_S of the group $\mathfrak{G}_2$ into $\mathfrak{G}_1$ is obviously a homomorphism. Consequently, the following proposition holds: if T is a solution of equation (1) in $\mathfrak{G}_2$, then the image T' of the sequence T under the mapping Φ_S is a solution of (1) in $\mathfrak{G}_1$.

Theorem 1. *Let a normal equation be given*

$$W(X_1, X_2, \dots, X_n) = 1, \tag{3}$$

a free group $\mathfrak{G}$ of rank $r \geq n$, a solution T of equation (3) in $\mathfrak{G}$, and a free group $\mathfrak{G}'$ of rank $r' = n - 1$. Then, whatever the sequence T may be, one can always indicate a solution S' of equation (3) in $\mathfrak{G}'$ and a mapping Φ_S of the free group $\mathfrak{G}'$ into $\mathfrak{G}$ such that the image of S' under the mapping Φ_S is T.

The proof of this theorem follows very simply from Lemmas 1 and 2.

Lemma 1. *If $T = \{T_1, T_2, \dots, T_n\}$ is a solution of equation (3) in $\mathfrak{G}$, and for every i, $T_i \neq 1$, then there exists a sequence $S = \{S_1, S_2, \dots, S_n\}$ such that T associates S and*

$$\sum_i |T_i^\delta| > \sum_i |S_i^\delta|.$$

Lemma 2. *Let a sequence $U = \{U_1, U_2, \dots, U_m\}$ of elements of the group X and numbers e, ε satisfying the relations*

$$e = \pm 1, \qquad \varepsilon = \pm 1.$$

be given.

Suppose that U contains a pair of terms U_i, U_j such that $U_i \neq U_j^{\pm 1}$. Define another sequence $U' = \{U'_1, U'_2, \dots, U'_m\}$ in the same group X as follows:

1. $U'_k = U_k$, if $U_k \neq U_i^{e'}$ $(e' = \pm 1)$.
2. $U'_k = (U_i^{\varepsilon} U_j^{\varepsilon})^{e'}$, if $U_k = U_i^{e'}$ $(k = 1, 2, \dots, m)$.

If

$$[U_1^\delta = [U_2^\delta = \dots = [U_m^\delta = 1,$$
$$\prod_i U_i = U_1 \cdot U_2 \cdots U_m,$$

then

$$\prod_i U'_i \neq 1.$$

A simple proof of Theorem 1 is obtained if, instead of Lemma 1, one uses in its expanded form Nielsen' s theorem that every subgroup of a free group of rank r, generated by elements $A_1, A_2, \dots, A_l$ from $\mathfrak{G}$, is isomorphic to some free group $\mathfrak{G}^*$ with generators $G_1^*, G_2^*, \dots, G_{l'}^*$, where $l' \leq l$ ([3]). However, proving Lemma 1 is considerably simpler than proving Nielsen' s theorem.

We shall next consider coefficient-free equations of the form

$$W(X, Y) = 1, \tag{4}$$

i.e. equations in only two unknowns. Moreover, we shall suppose that every equation of the form (4) considered is normal and satisfies the condition

$$W(X, Y) = \prod_k (X^{c_k} Y^{d_k}), \tag{5}$$

where $c_k d_k \neq 0$.

From Theorem 1 it is not hard to derive the following result.

Corollary 1. *A sequence $S = \{S_1, S_2\}$ of elements of the free group $\mathfrak{G}$ of rank r is a solution of equation (4) in $\mathfrak{G}$ if and only if $S_1 = A^m$, $S_1 = A^n$, where m, n are a solution of equation (6); A is an arbitrary element in $\mathfrak{G}$.*

The proposition just formulated makes it possible to prove a more general result, whose formulation requires additional terms.

Let there be given a system of coefficient-free equations Ω in the unknowns $X_1, X_2, \dots, X_n$ such that each equation of the system Ω has the form

$$W(X_i, X_j) = 1 \quad (i \neq j,\ 1 \leq i, j \leq n).$$

We shall henceforth denote by σ an arbitrary list of unknowns of the system Ω, assuming, moreover, that σ need not contain every unknown of the system Ω; in other words, the list σ need not be complete. In what follows we shall speak only of such lists σ that contain more than one unknown.

Definition 3. Fix a list σ of unknowns of the system Ω and define a relation R for the unknowns of the list σ as follows:

1. X_i is in the relation R to X_i for every X_i in σ (reflexivity).
2. X_i is in the relation R to X_j if there is a sequence of unknowns $X_{j_1}, X_{j_2}, \dots, X_{j_l}$ of the list σ such that $X_{j_1} = X_i$, $X_{j_l} = X_j$, and for every k, $1 \le k \le l$, there is in the system Ω an equation in the unknowns $X_{j_k}, X_{j_{k+1}}$ $(j_k \neq j_{k+1})$.

The relation R, obviously, determines a partition of the unknowns of the list σ into classes, since it is reflexive, symmetric, and transitive.

Definition 4. Let a_i be integers, t_i integer variables, and Γ_i variables with values in a free group G of finite rank. We shall call a sequence of expressions

$$\Gamma_1^{a_1 t_1},\ \Gamma_2^{a_2 t_2},\ \dots,\ \Gamma_n^{a_n t_n}$$

a **biparametric sequence of type** σ, if the following conditions are satisfied:

1. $a_i = 0$ if the unknown X_i of the system Ω does not belong to the list σ.
2. In the case when the unknowns X_i and X_j belong to the list σ, the variables denoted by the letters t_i and t_j coincide if and only if X_i is in the relation R to X_j.
3. In the case when the unknowns X_i and X_j belong to the list σ, the variables denoted by the letters Γ_i and Γ_j coincide if and only if X_i is in the relation R to X_j.

We shall call a sequence $S = \{S_1, S_2, \dots, S_n\}$ of elements of the free group G a **value of the biparametric sequence**

$$\Gamma_1^{a_1 t_1},\ \Gamma_2^{a_2 t_2},\ \dots,\ \Gamma_n^{a_n t_n}$$

of type σ, if there are sets of values of the variables Γ_i and t_i $(i = 1, 2, \dots, n)$ such that, for every i $(i = 1, 2, \dots, n)$,

$$S_i = C_i^{a_i m_i},$$

where C_i and m_i are the values of the variables Γ_i and t_i from these sets.

Theorem 2. For every system Ω and every free group G, one can construct a finite number of biparametric sequences of different types σ such that a sequence T of elements of the group G satisfies Ω if and only if T is a value of one of these biparametric sequences.

Theorem 2 also holds for systems of equations in words, when the concepts “biparametric sequence of type σ” and “system of equations Ω” are replaced by the corresponding equivalents.

Institute of Electronics and Computer Engineering
Academy of Sciences of the Latvian SSR

Received
3 January 1964

CITED LITERATURE

1. D. Skordew, B. E. Sendow, *Zs. math. Logik u. Grundl. Math.*, **7**, No. 4, 289 (1961).

2. A. A. Lorents, DAN, **148**, No. 6, 1253 (1963).

3. M. Hall, *Theory of Groups*, IL, 1962.

Note: Figure translations are in progress. See original paper for figures.