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Abstract

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FUNCTORS IN CATEGORIES OF BANACH SPACES DEFINED BY *KB*-LINEALS

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Functors in categories of Banach spaces were considered in the papers of A. S. Shvarts (^{1}) and B. S. Mityagin and A. S. Shvarts (^{2}). In these works a theory of duality of functors in the category of Banach spaces was constructed; in particular, functors defined by normed ideals of sequences and of measurable functions were studied.

In the present note, on the tensor product $E \otimes X$ of a KB-lineal E and a Banach space X, a natural crossnorm is defined, and the functor Φ_E is studied, assigning to each X the completion of $E \otimes X$ with respect to this crossnorm. The functor Φ_E includes, as special cases, functors defined by minimal normed ideals of sequences and of measurable functions, as well as other concrete functors. A realization of the space $\Phi_E(X')$, the space of mappings $\{\Phi_E \to \Phi_F\}$, and the dual functor $D\Phi_E$ are described. As consequences, answers are obtained to some questions posed in (^{2}), and a theorem on the nuclearity of one class of composite mappings.

A **crossnorm** on the tensor product $X \otimes Y$ of Banach spaces X, Y is a norm p satisfying the conditions: $p(x \otimes y) = \|x\|_X \|y\|_Y$ for all $x \in X$, $y \in Y$, and $p'(x' \otimes y') = \|x'\|_{X'} \|y'\|_{Y'}$ for all $x' \in X'$, $y' \in Y'$, where p' is the norm on $(X \otimes_p Y)'$ conjugate to p, and $x' \otimes y'$ is the functional on $X \otimes_p Y$ acting by the formula

$$\left\langle \sum_{k=1}^{n} x_k \otimes y_k,\ x' \otimes y' \right\rangle = \sum_{k=1}^{n} \langle x_k, x' \rangle \langle y_k, y' \rangle. \tag{3}$$

Among crossnorms there exist the maximal π and the minimal ε (^{3},^{4}). The completions of $X \otimes Y$ with respect to the norms π and ε are denoted by $X \widehat{\otimes} Y$ and $X \overset{\vee}{\otimes} Y$.

Let $\mathcal{F}$ be a functor of type Σ in some category $\mathcal{K}$ of Banach spaces.* It can be shown that for every $X \in \mathcal{K}$, $\mathcal{F}(X)$ is isometric to the completion of $\mathcal{F}(I) \otimes X$ with respect to some crossnorm, and for $\alpha \in (X \to Y)$

$$\mathcal{F}(\alpha) = 1_{F(I)} \otimes \alpha$$

and for $\varphi \in \{\mathcal{F} \to \mathcal{G}\}$, where $\mathcal{G}$ is a functor in $\mathcal{K}$,

$$\varphi_X = \varphi_I \otimes 1_X.$$

The identity mapping of $\mathcal{F}(I) \otimes X$ onto itself extends to a continuous linear mapping of $\mathcal{F}(X)$ into $\mathcal{F}(I)\widehat{\otimes}X$; we shall say that the functor $\mathcal{F}$ **satisfies the condition of mutual uniqueness** if this mapping is one-to-one for every $X \in \mathcal{K}$.

Let E be a KB-lineal (^{5}), and X a Banach space. Introduce on $E \otimes X$ the norm n_E, putting, for $z = \sum_{k=1}^n e_k \otimes x_k$,

$$n_E(z) = \inf\left\{ \|u\|_E : \; u \geq \left| \sum_{k=1}^n e_k \langle x_k, x' \rangle \right| \text{ for all } x', \; \|x'\| \leq 1 \right\}.$$

It is verified directly that the function Φ_E, assigning to each $X \in \mathcal{K}$ the completion of $E \otimes X$ with respect to the norm n_E, and to each $\alpha \in (X \to Y)$ the mapping

$$\Phi_E(\alpha) = 1_E \otimes \alpha,$$

is a functor in any category $\mathcal{K}$.

Theorem 1**. *If E is, respectively, the minimal normed ideal of sequences n, the corresponding ideal of measurable func-*

* Concerning the definitions and notation relating to functors in categories of Banach spaces, see (^{1},^{2}).

** For definitions of normed ideals of sequences and measurable functions and of the functors defined by them, as well as of the functor C, see (^{2}).

$\cdots N$, the space C of continuous functions on the segment $[a, b]$, then the functor Φ_E is, respectively, n, N, C.

Proof. In the cases under consideration there exists

$$u_0 = \sup_{\|x'\| \leqslant 1} \left| \sum_{k=1}^n e_k \langle x_k, x' \rangle \right| \in E \quad \text{for every} \quad z = \sum_{k=1}^n e_k \otimes x_k \quad \text{and} \quad n_E(z) =$$

$$= \|u_0\|_E.$$

Further, since the supremum in a KB-lineal of sequences (functions) is determined coordinatewise (pointwise), the norm n_E coincides on $E \otimes X$ with the norm of the corresponding space $E(X)$, and the assertion of the theorem follows from the fact that $E \otimes X$ is dense in $E(X)$ ($E = n, N, C$).

A mapping $a \in (X \to E)$ that carries the unit ball of X into a set E bounded in the sense of the ordering (see (^{5})) will be called **proper***. On the space of proper mappings we introduce the norm

$$n(a) = \inf\{\|u\|_E : u \geqslant |ax| \quad \text{for all } x, \; \|x\| \leqslant 1\}.$$

We shall say that a KB-lineal E **satisfies condition** $(*)$ if, for every monotonically increasing sequence $0 \leqslant e_n \in E$ unbounded above, $\|e_n\| \to \infty$ as $n \to \infty$. All KB-spaces $(^5)$ (in particular l^p and L^p) and the space C satisfy condition $(*)$.

Theorem 2. *If E satisfies condition $(*)$, and Φ_E satisfies the condition of mutual uniqueness, then $\Phi_E(X')$ is isometric to the closure of the set of continuous finite-dimensional mappings of X into E in the space of proper mappings of X into E. If, moreover, there exists a sequence of finite-dimensional operators $v_n \in (E \to E)$ such that $v_n e \to e$ as $n \to \infty$ and $v_n e \leqslant e$ for all $0 \leqslant e \in E$, then $\Phi_E(X')$ is isometric to the space of all proper mappings of X into E.*

Remark 1. Φ_E satisfies the condition of mutual uniqueness for $E = n, N, C$.

Remark 2. A sequence v_n with the required properties exists for every space of sequences with the usual ordering, in which the "orts" $\delta_n = (0, \dots, 0, \underset{n}{1}, 0, \dots)$ form a basis.

Let us outline the proof. According to the condition of mutual uniqueness, each element $z \in \Phi_E(X')$ may be regarded as a completely continuous mapping of X into E. Using condition $(*)$, one can show that this mapping is proper. Since the norm n_E of the element $z = \sum_{k=1}^{n} e_k \otimes x'_k$ is equal to the norm n of the corresponding mapping $\sum_{k=1}^{n} \langle \cdot\,, x'_k \rangle e_k$, the first part of the theorem follows. Let a be a proper mapping of X into E and $|ax| \leqslant e_0$ for all $x \in X$, $\|x\| \leqslant 1$. Then $|(1_E - v_n)ax| \leqslant (1_E - v_n)e_0$ for all $x \in X$, $\|x\| \leqslant 1$, and

$$n(a - v_{na}) = \inf\{\|u\| : u \geqslant |(1_E - v_n)ax| \text{ for all } x;\ \|x\| \leqslant 1\} \leqslant \|(1_E - v_n)e_0\|,$$

so that a is approximated in the norm n by finite-dimensional mappings $v_n a$, and, consequently, $a \in \Phi_E(X')$.

Let E, F be KB-lineals. A mapping $a \in (E \to F)$ is called **regular** if it can be represented as the difference of two positive mappings (see $(^5)$).

We shall say that a KB-lineal E **satisfies condition** $(**)$ if, for every monotonically increasing sequence $0 \leqslant e_n \in E$ possessing a supremum $\sup_n e_n$,

$$\|e_n\| \to \left\|\sup_n e_n\right\|$$

as $n \to \infty$. All KB-spaces, normalized ideals of sequences and measurable functions, and the space C satisfy condition $(**)$.

* When E is an $\mathcal{K}^+$-space, proper mappings coincide with bounded mappings (see $(^5)$, p. 246).

Theorem 3. Let E be a KB-lineal, and let F be a conditionally complete KB-lineal (see $(^5)$), satisfying conditions $(*)$ and $(**)$, and suppose that either

E or F is separable. Then the space of mappings $\{\Phi_E \to \Phi_F\}$ is isometric to the Banach space $(E \to F)_r$ of regular mappings of E into F with norm $\nu(\alpha) = \| |\alpha| \|_{(E\to F)}$; under this isometry the mapping $\alpha \in (E \to F)_r$ corresponds to the class of mappings $\alpha \otimes 1_X \in (\Phi_E(X) \to \Phi_F(X))$, $X \in \mathfrak{K}$. The assertion of the theorem is valid in every category $\mathfrak{K}$ containing the space c_0.

Remark 1. Theorem 3 remains valid if, instead of separability of E or F, one requires that every set in E bounded in the order sense be separable.

Remark 2. For the case $E = F = L^p$, Theorem 3 gives a positive answer to the question about the ring of operators in the functor L^p, posed in (^{2}).

Lemma. Let E, F be KB-lineals. To a regular mapping $\alpha : E \to F$ there corresponds a mapping of functors $\Phi_E \to \Phi_F$ in every category $\mathfrak{K}$, and

$$\|\alpha\|_{\{\Phi_E\to\Phi_F\}} \le \nu(\alpha).$$

Proof of Theorem 3. Suppose $\alpha \in (E \to F)$ is not regular. Then in E there is a set Q, bounded in the order sense, such that αQ is not bounded in the same sense (see (^{5}), p. 232). Take a sequence of elements $e_n \in Q$ whose images αe_n are dense in αQ. Then the sequence

$$g_n = \sup_{1\le k\le n} |\alpha e_k|$$

is not bounded in F in the order sense; applying condition $(*)$, we obtain that $\|g_n\| \to \infty$ as $n \to \infty$. Consider the elements

$$z_n = \sum_{k=1}^{n} e_k \otimes \delta_k \in \Phi_E(c_0),$$

where

$$\delta_k = (0, \dots, 0, 1, 0, \dots).$$

We have

$$\|z_n\|_{\Phi_E(c_0)} = \left\| \sup_{1\le k\le n} |e_k| \right\|_E \le \|e_0\|_E,$$

where e_0 is an element of E majorizing the moduli of the elements of Q. On the other hand,

$$\|(\alpha\otimes 1_{c_0}) z_n\|_{\Phi_F(c_0)} = \left\| \sup_{1\le k\le n} |\alpha e_k| \right\|_F = \|g_n\|_F \to \infty$$

as $n \to \infty$, so that

$$\alpha \notin \{\Phi_E \to \Phi_F\}$$

if $\mathfrak{K} \ni c_0$. Hence, and from the lemma, it follows that

$$(E \to F)_r = \{\Phi_E \to \Phi_F\}.$$

Now let $\alpha \in (E \to F)_r$. Fix $0 \leq e_0 \in E$ and take a sequence of elements

$$e_n \in Q_{e_0} = \{e : |e| \leq e_0\},$$

whose images αe_n are dense in αQ_{e_0}. Put

$$u_n = \sup_{1 \leq k \leq n} |\alpha e_k|.$$

Then

$$\sup_{|e| \leq e_0} |\alpha e| = \sup_n u_n,$$

and, by $(**)$,

$$\left\| \sup_{|e| \leq e_0} |\alpha e| \right\| = \lim_{n \to \infty} \|u_n\|.$$

Moreover,

$$\|u_n\|_F = \left\| \sum_{k=1}^{n} \alpha e_k \otimes \delta_k \right\|_{\Phi_F(c_0)} \leq \|\alpha\|_{\{\Phi_E \to \Phi_F\}} \left\| \sum_{k=1}^{n} e_k \otimes \delta_k \right\|_{\Phi_E(c_0)}$$

$$= \|\alpha\|_{\{\Phi_E \to \Phi_F\}} \sup_{1 \leq k \leq n} |e_k| \leq \|\alpha\|_{\{\Phi_E \to \Phi_F\}} \|e_0\|.$$

Passing to the limit as $n \to \infty$, we obtain

$$\left\| \sup_{|e| \leq e_0} |\alpha e| \right\| \leq \|\alpha\|_{\{\Phi_E \to \Phi_F\}} \|e_0\|.$$

Further, since

$$|\alpha| e_0 = \sup_{|e| \leq e_0} |\alpha e|$$

(see (^{5}), p. 231),

$$\nu(\alpha) = \sup_{0 \leq e_0 \in E} \frac{\||\alpha| e_0\|}{\|e_0\|} = \sup_{0 \leq e_0 \in E} \frac{\left\| \sup_{|e| \leq e_0} |\alpha e| \right\|}{\|e_0\|} \leq \|\alpha\|_{\{\Phi_E \to \Phi_F\}}.$$

Thus

$$\nu(\alpha) = \|\alpha\|_{\{\Phi_E \to \Phi_F\}},$$

and the theorem is proved.

We shall say that a KB-lineal E **satisfies condition** $(***)$ if, for every Banach space X and every $z \in E \otimes X$,

$$n_E(z) = \inf \left\| \sum_{k=1}^{n} |e_k| \, \|x_k\| \right\|,$$

where the infimum is taken over all representations

$$z = \sum_{k=1}^{n} e_k \otimes x_k.$$

It can be shown that normed ideals

spaces of sequences and measurable functions, and the space C, satisfy condition $(***)$; this condition is satisfied by every KB-lineal of functions E with the usual ordering, containing the step functions as a dense subspace.

Call a mapping $\alpha : E \to X$ **summing** if it carries every series $\sum_{k=1}^{\infty} e_k$ in E for which the series $\sum_{k=1}^{\infty} |e_k|$ converges into an absolutely convergent series* $\sum_{k=1}^{\infty} \alpha e_k$ in X. On the space of summing mappings consider the norm

$$v_E(\alpha) = \sup \sum_{k=1}^{n} \|\alpha e_k\|_X \Bigg/ \left\| \sum_{k=1}^{n} |e_k| \right\|_E ,$$

where the supremum is taken over all possible finite sets of elements $e_k \in E$, $k = 1, \ldots, n$.

Theorem 4. Let E be a KB-lineal satisfying condition $(***)$. Then $D\Phi_E(X)$ is isometric to the space of summing mappings of E into X in every category $\mathfrak{K}$ containing the space l^1.

Remark. In (2) this theorem was proved for $E = C$.

Let $B(E)$ be the space of sequences $(e_k) \subset E$ for which the series $\sum_{k=1}^{\infty} |e_k|$ converges, with norm

$$\|(e_k)\| = \left\| \sum_{k=1}^{\infty} |e_k| \right\|_E .$$

Lemma 1. $B(E)$ is complete for every KB-lineal E.

Let α be a summing mapping from E into X. Consider the mapping $\Psi_\alpha : B(E) \to l^1(X)$, defined by the formula $\Psi_\alpha(e_k) = (\alpha e_k)$.

Lemma 2. Ψ_α is continuous for every KB-lineal E.

Lemma 3. Let E be a KB-lineal satisfying condition $(***)$, and let α be a summing mapping from E into X. Then α generates an element of $D\Phi_E(X)$ in every category $\mathfrak{K}$, and

$$\|\alpha\|_{D\Phi_E(X)} \leq v_E(\alpha).$$

Lemma 4. Let E be a KB-lineal and let $\alpha : E \to X$ generate an element of $D\Phi_E(X)$ in a category $\mathfrak{K}$ containing l^1. Then α is a summing mapping and

$$v_E(\alpha) \leq \|\alpha\|_{D\Phi_E(X)}.$$

Theorem 4 follows from Lemmas 3 and 4; Lemma 3 rests on Lemmas 1 and 2.

In (2) the problem was posed of describing the functor DN in the category of all Banach spaces, and the conjecture was made that $DN = N'$.** This conjecture is not true, since the embedding $N \subset L^1$ determines an element of $DN(L^1)$ not belonging to $N'(L^1)$. For the case of a minimally normed ideal, Theorem 4 gives a description of the functor DN.

Theorem 5. Let X, Y be Banach spaces, E a KB-lineal satisfying conditions $(*)$ and $(***)$, Φ_E satisfy the condition of mutual one-to-one correspondence, $u : X \to E$ a proper mapping approximable by finite-dimensional mappings in norm n, and $\alpha : E \to Y$ a summing mapping. Then their product αu is a nuclear mapping from X into Y.

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[*] A series $\sum_{k=1}^{\infty} x_k$ in a Banach space X is called **absolutely convergent** if

$$\sum_{k=1}^{\infty} \|x_k\| < \infty.$$

[**] N' denotes the associated ideal (2).

Note: Figure translations are in progress. See original paper for figures.

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