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Abstract

Full Text

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ON LOWER ESTIMATES FOR ENTIRE FUNCTIONS OF FINITE ORDER AND ON THE CONVERGENCE OF DIRICHLET POLYNOMIALS

(Presented by Academician I. M. Vinogradov on 10 XII 1964)

In a number of questions of complex analysis (see ([1])) one encounters the following problem: let $f(z)$ be an entire function of refined order $\rho(t)$, whose zeros, lying in the angle $|\arg z| < \alpha$ ($\alpha > 0$), cluster at the positive axis: $\arg \lambda_i \to 0$ ($i \to \infty$). In terms of the distribution of the points λ_i, find conditions under which

$$\varliminf_{\theta \to 0} H(\theta) > -\infty, \tag{1}$$

where

$$H(\theta) = \varliminf_{r \to \infty} \frac{\ln |f(re^{(\theta)})|}{r^{\rho(r)}}.$$

Denote by $n_\sigma(z)$ the number of points λ_i in the disk $\{h: \ |h - z| \le \sigma |z|\}$. Then the following assertion holds:

Theorem 1. *In order that (1) hold, it is necessary and sufficient that*

$$\lim_{\varepsilon \to 0} \varlimsup_{x \to \infty} \frac{1}{x^{\rho(x)}} \int_\varepsilon^1 \frac{n_\sigma(x)}{\sigma} \, d\sigma < \infty.$$

The indicated condition is more general than the requirement of finiteness of the maximal density

$$D = \lim_{\varepsilon \to 0} \varlimsup_{r \to \infty} \frac{1}{r^{\rho(r)}} \frac{n[r(1+\varepsilon)] - n[r(1-\varepsilon)]}{2\varepsilon},$$

where $n[r]$ is the number of points λ_i whose moduli do not exceed r. Therefore the class of sequences λ_i $(i = 1, 2, ...)$ for which (1) holds is broader than the class of sequences with finite maximal density.

The proof of Theorem 1 is based on the following representation for the logarithm of the modulus of an entire function of refined order $\rho(t)$:

$$\ln|f(z)| = -\int_0^1 \frac{n_\sigma(z)}{\sigma}\, d\sigma + O(1)|z|^{\rho(|z|)} \quad (z \neq 0),$$

where $O(1)$ denotes a function bounded outside any disk with center at the origin, and the integral is understood in the improper sense:

$$\lim_{\varepsilon\to 0}\int_\varepsilon^1 \frac{n_\sigma(z)}{\sigma}\, d\sigma.$$

A special case of the theorem, obtained for $\rho(t) \equiv 1$, makes it possible to give a definitive answer to a question on the convergence of Dirichlet polynomials posed by A. F. Leont' ev ([1]):

Let G be a convex domain, and let the sequence of points λ_i $(i = 1, 2, ...)$ cluster to the real axis: $\arg\lambda_i \to 0$ $(i \to \infty)$, and let the system $\{e^{-\lambda_i z}\}$ be incomplete in G. We shall say that the convergence of Dirichlet polynomials extends inside the vertical half-plane $\operatorname{Re} z > a$ if every sequence of Dirichlet polynomials

$$P_n(z) = \sum_{i=1}^{p_n} a_i^{(n)} e^{-\lambda_i z} \quad (n = 1, 2, ...), \tag{2}$$

converging uniformly inside G, will converge uniformly inside the half-plane $\operatorname{Re} z > a$ (the same half-plane for all sequences (2)). The question arises: under what conditions imposed on the sequence λ_i $(i = 1, 2, ...)$ does the convergence of Dirichlet polynomials extend inside a vertical half-plane?

An exhaustive answer to this question is given by

Theorem 2. *In order that the convergence of the Dirichlet polynomials (2) extend inside some vertical half-plane* $\operatorname{Re} z > a$*, it is necessary and sufficient that*

$$\lim_{\varepsilon\to 0}\overline{\lim}_{x\to\infty}\frac{1}{x}\int_\varepsilon^1 \frac{n_\sigma(x)}{\sigma}\, d\sigma < \infty. \tag{3}$$

This condition, in particular, will be satisfied if the sequence λ_i $(i = 1, 2, ...)$ is measurable: $n/|\lambda_i| \to \sigma$. Therefore Theorem 2 contains the corresponding result obtained earlier by A. F. Leont' ev (see ([2,3])) and applying only to measurable sequences.

The proof of Theorem 2 can be carried out according to the general scheme developed by J. P. Kahane ([4]). In doing so, one has to overcome the difficulty connected with the construction of an entire function of exponential type $\psi(\lambda)$ with the properties: 1) the conjugate diagram of $\psi(\lambda)$ is contained in G; 2) inside some angle $|\arg z| < \alpha$ $(\alpha > 0)$, the function $\psi(\lambda)$ vanishes only at the points λ_i contained in this angle.

If one restricts oneself to proving only the sufficiency of condition (3), then the difficulty indicated above can be avoided by basing the proof on the direct representation for Dirichlet polynomials through the function $\omega(\mu, P)$, introduced by A. F. Leont' ev ([5]).

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Note: Figure translations are in progress. See original paper for figures.