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Abstract

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On the Stability of Equilibrium States of Nonholonomic Systems

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In the present paper it is shown that a nonholonomic system possesses the special feature that its equilibrium states cannot be isolated, but form a manifold whose dimension is equal to the number of equations of the nonholonomic constraints. This feature also accounts for the presence of zero roots in the characteristic equation*. A theorem on the asymptotic stability of a manifold of equilibrium states is formulated. The theory set forth is illustrated by an example.

1. Let a system with Lagrange function $L = L(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n)$ and generalized forces $Q_\beta = Q_\beta(q, \dot{q})$ $(\beta = 1, 2, \dots, n)$ be subject to nonholonomic constraints defined by m $(m < n)$ equations**:

$$\omega_{\alpha\beta}(q_1, \dots, q_n)\dot{q}_\beta = 0 \qquad (\alpha = 1, 2, \dots, m;\ \beta = 1, 2, \dots, n). \tag{1}$$

The equations of motion with undetermined multipliers

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\beta} - \frac{\partial L}{\partial q_\beta} = Q_\beta + \lambda_\alpha \omega_{\alpha\beta} \tag{2}$$

together with (1) form a system of $(n+m)$ equations for $q_1, q_2, \dots, q_n, \lambda_1, \lambda_2, \dots, \lambda_m$. From (1) and (2) it follows that the equilibrium states of a nonholonomic system are determined by n equations

$$\frac{\partial L}{\partial q_\beta} + Q_\beta + \lambda_\alpha \omega_{\alpha\beta} = 0 \tag{3}$$

with respect to the $(n + m)$ unknowns $q_1, \dots, q_n, \lambda_1, \dots, \lambda_m$. Consequently, in the general case we have a manifold of equilibrium states, forming in the n-dimensional configuration space a surface O_m of dimension m. Indeed, expressing, by means of equations (3), the generalized coordinates $q_1, \dots, q_n$ in terms of $\lambda_1, \dots, \lambda_m$, we obtain a surface in parametric representation $q^0_\beta = q^0_\beta(\lambda_1, \dots, \lambda_m)$. We note that in particular problems there may occur a case in which equations (3) are not independent, which entails an increase in the dimension of the manifold of equilibrium states.

Fig. 1

Figure 1: Fig. 1

* The question of the stability of equilibrium states of nonholonomic systems was considered in the works of Whittaker and others ([1–5]). The methods of investigating stability proposed in these works, as well as the viewpoints on the nature of the zero roots, do not agree with one another. As is known, Whittaker considered it possible to integrate the linearized equations of nonholonomic constraints, after which the distinction between holonomic and nonholonomic systems disappeared. Bottema, having removed the inaccuracy in Whittaker' s reasoning, showed that, unlike a holonomic system, the characteristic determinant of a nonholonomic system is nonsymmetric and that the characteristic equation has zero roots. Bottema concluded that this is a critical case in the theory of stability of an isolated equilibrium state. M. A. Aizerman and F. R. Gantmacher showed that this case reduces to the special case studied by A. M. Lyapunov and I. G. Malkin.

** Here and below, summation is to be performed over repeated indices, and a dot over a letter denotes differentiation with respect to time.

2. Eliminating from equations (2) the undetermined multipliers $\lambda_1, \dots, \lambda_m$, we write the equations of motion (1), (2) of the nonholonomic system in normal form

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_{2n-m}) \quad (i = 1, 2, \dots, 2n-m), \tag{4}$$

where by x_i we denote the variables $q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_{n-m}$. Suppose that in the phase space $(x_1, x_2, \dots, x_{2n-m})$ the surface O_m is defined by the equations

$$x_i^0 = x_i^0(u_1, \dots, u_m) \quad (i = 1, 2, \dots, 2n-m).$$

Along with the variables $u_1, \dots, u_m$, introduce new variables $v_1, \dots, v_{2(n-m)}$ by means of the relations

$$x_i = x_i^0(u_1, \dots, u_m) + \gamma_{ij}(u_1, \dots, u_m) v_j.$$

Fig. 1

In the new variables, equations (4) will be written in the form

$$\dot{u}_i = g_i(u, v), \qquad \dot{v}_j = g_j(u, v). \tag{5}$$

We linearize the equations of motion (5) in a neighborhood of the surface O_m of equilibrium states. Expanding the right-hand sides of equations (5) in a series in the small quantities v, we obtain:

$$\begin{aligned} \dot{u}_i &= a_i(u_1, \dots, u_m) + a_{ij}(u_1, \dots, u_m)v_j + O(\|v\|^2) + \cdots, \\ \dot{v}_j &= b_j(u_1, \dots, u_m) + b_{jk}(u_1, \dots, u_m)v_k + O(\|v\|^2) + \cdots, \end{aligned} \tag{6}$$

where a_i and b_j are equal to zero, since $v = \dot{v} = \dot{u} = 0$ on the surface O_m. Accordingly, the characteristic equation of system (6) for any point of the surface O_m has the form

$$p^m \left| b_{ij} - p\delta_{ij} \right| = 0. \tag{7}$$

3. Let us pass to determining the stability conditions. It follows from the preceding that studying the stability of equilibrium states of a nonholonomic system is meaningful only with respect to small deviations from the surface O_m. In this case it is natural to consider the second group of equations (6) independently of the first group, treating temporarily the variables $u_1, \dots, u_m$ as parameters. The characteristic polynomial of this auxiliary system differs from determinant (7) only by the absence of the factor p^m.

Suppose that in some domain G of values of $u_1, \dots, u_m$ the equilibrium state $v = 0$ of the system of equations

$$\dot{v}_j = b_{jk}(u_1, \dots, u_m)v_k \tag{8}$$

is asymptotically stable, so that * $\|v\| < M\|v^0\|e^{-\sigma t}$, where $\sigma > 0$, and v^0 are the initial values of the variables v; then the following holds.

Theorem. Let the initial values $u_1^0, \dots, u_m^0,\ v_1^0, \dots, v_{2(n-m)}^0$ be such that the values $u_1^0, \dots, u_m^0$ lie inside the domain G of asymptotic stability, and $v_1^0, \dots, v_{2(n-m)}^0$ are sufficiently small; then, by virtue of equations (5),

$$\lim_{t\to+\infty} v_j(t) = 0, \qquad \lim_{t\to+\infty} u_i(t) = u_i^*, \tag{9}$$

* The symbol v denotes the vector with components v_j, and the symbol $\|v\| = \left[v_1^2 + \cdots + v_{2(n-m)}^2\right]^{1/2}$.

where $u_i^* \in O_m$, but, in general, $u_i^* \neq u_i^0$. In this case, for the variables $v_j(t)$ the estimate

$$\|v(t)\| < M'\|v^0\|e^{-\sigma' t}, \tag{10}$$

holds, where $\sigma \gg \sigma' > 0$.

Proof. Let us write equations (6) in the form

$$\dot{u}_i = \{a_{ij}(u_1, \dots, u_m) + \Delta a_{ij}\}\, v_j, \tag{11}$$

$$\dot{v}_j = \{b_{jk}(u_1, \dots, u_m) + \Delta b_{jk}\}\, v_k,$$

where $|\Delta a_{ij}| < \varepsilon$ and $|\Delta b_{jk}| < \varepsilon$, if the condition

$$\|u - u^0\| < \delta(\varepsilon), \qquad \|v\| < \delta(\varepsilon) \quad [\delta(\varepsilon) > 0]. \tag{12}$$

is satisfied.

As long as the inequalities (12) are satisfied, for sufficiently small $\delta = \delta^*$, for the solution of equations (11), the estimate (6) holds:

$$\|v(t)\| < M'\|v^0\|e^{-\sigma' t}, \tag{13}$$

and therefore

$$\|\dot{u}\| < N\|v^0\|e^{-\sigma' t},$$

$$\|u(t)\| < \frac{N}{\sigma}\|v^0\|. \tag{14}$$

Suppose that, for the chosen value $\delta = \delta^*$, the inequality

$$\|v^0\| < \min\left(\frac{\delta^*}{2M'}, \frac{\delta^*\sigma}{2N}, \frac{\delta^*}{2}\right) \tag{15}$$

is satisfied.

At the initial instant $t = 0$, conditions (12) are satisfied for $\delta = \delta^*/2$; therefore, by virtue of the continuous dependence of the solution on the time t, they will be satisfied over some time interval Δt_0, and consequently, over this interval the estimates (13) and (14) will hold. After the time Δt_0, by virtue of these estimates and inequality (15), the quantity $v(\Delta t_0)$ satisfies inequalities (12) with $\delta = \delta^*/2$, and therefore these inequalities are satisfied over some time interval $\Delta t_0 + \Delta t_1$. Continuing these arguments, we establish that estimates (13) and (14) hold for all t, since $\Delta t_s > \tau > 0$, because at the instant of time $\Delta t_0 + \dots + \Delta t_{s-1}$ the inequalities (13) and (14) hold. From the validity of estimates (13) and (14) for all t, the assertion of the theorem follows.

Fig. 2

Figure 2: Fig. 2

As an example, consider the motion of an axisymmetric body, bounded below by a spherical surface of radius R, which can roll without slipping in a spherical cup of radius R_1. The center of mass of the body is located at a distance l from the center of its spherical surface. In accordance with Fig. 1, the generalized coordinates of the body are the angles $\theta, \psi, \varphi, \theta_1, \psi_1$. The Lagrangian function is

Fig. 2

$$\begin{aligned} L = &\frac{1}{2} m(R_1 - R)^2(\dot\theta_1 + \dot\psi_1 \sin^2\theta_1) + \frac{1}{2}(A + ml^2)(\dot\theta^2 + \dot\psi^2 \sin^2\theta) \\ &+ ml(R_1 - R)\{\dot\psi\dot\psi_1 \sin\theta \sin\theta_1 \cos(\psi - \psi_1) + \dot\theta\dot\theta_1[\cos\theta\cos\theta_1\cos(\psi - \psi_1) \\ &\quad + \sin\theta\sin\theta_1] + \dot\theta\dot\psi_1\cos\theta\sin\theta_1\sin(\psi - \psi_1) - \dot\psi\dot\theta_1\sin\theta\cos\theta_1\sin(\psi - \psi_1)\} \\ &+ \frac{1}{2}C(\dot\varphi + \dot\psi\cos\theta)^2 + mg[(R_1 - R)\cos\theta_1 + l\cos\theta], \end{aligned}$$

where m is the mass, A, C are the principal moments of inertia, and g is the acceleration of gravity. The dissipation function is

$$\Phi = \frac{1}{2}h\left(\frac{R_1}{R} - 1\right)^2(\dot\theta_1^2 + \dot\psi_1^2\sin^2\theta_1) + \frac{1}{2}h_1\{\dot\psi\cos\theta_1 - \dot\theta\sin\theta_1\sin(\psi - \psi_1) +$$

$$+\dot\varphi[\sin\theta\sin\theta_1\cos(\psi - \psi_1) + \cos\theta\cos\theta_1]\}^2,$$

where $h \geqslant 0$, $h_1 \geqslant 0$ are the coefficients of viscous rolling and spinning friction. The condition of rolling without slipping leads to two equations of nonholonomic constraints

$$(R_1 - R)\dot\psi_1\sin\theta_1 + R\dot\theta\cos\theta_1\sin(\psi - \psi_1) + R\dot\psi\sin\theta_1 +$$

$$+R\dot\varphi[\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1\cos(\psi - \psi_1)] = 0;$$

$$(R_1 - R)\dot\theta_1 + R\dot\theta\cos(\psi - \psi_1) + R\dot\varphi\sin\theta\sin(\psi - \psi_1) = 0.$$

Forming the equations of motion (2), where the generalized forces will be $Q_\beta = -\partial\Phi/\partial\dot q_\beta$, we then obtain the following equilibrium equations:

$$\lambda_1 \sin\theta_1 = 0; \qquad mg\sin\theta_1 = \lambda_2; \qquad \lambda_1 \sin\theta_1 = 0;$$
$$mgl\sin\theta = R\lambda_1 \cos\theta_1 \sin(\psi - \psi_1) + R\lambda_2 \cos(\psi - \psi_1); \tag{16}$$
$$\lambda_1[\cos\theta\sin\theta_1 - \sin\theta\cos\theta_1\cos(\psi-\psi_1)] + \lambda_2 \sin\theta\sin(\psi-\psi_1) = 0,$$

where λ_1, λ_2 are undetermined multipliers. From equations (16) it follows that the surface O_m of equilibrium states is determined by the equations $\psi = \psi_1$, $l\sin\theta = R\sin\theta_1$, and is three-dimensional. The characteristic equation is reduced to the form

$$p^3(a_0p^2 + a_1p + a_2)(b_0p^3 + b_1p^2 + b_2p + b_3) = 0,$$

whose coefficients denote certain functions of the system parameters. For values $R_1 > R > l > 0$, the stability region of the manifold of equilibrium states of the system is determined by the inequalities $b_3 > 0$, $b_1b_2 - b_0b_3 > 0$. In Fig. 2 the boundaries of the stability region are shown for $R_1 = 4R$ for a homogeneous hemispherical shell, where $\delta = h(mR\sqrt{gR})^{-1}$, $\delta_1 = h_1(mR\sqrt{gR})^{-1}$.

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Note: Figure translations are in progress. See original paper for figures.