INTEGRAL REPRESENTATION OF FUNCTIONS FORMING A MARKOV SYSTEM

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Abstract

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INTEGRAL REPRESENTATION OF FUNCTIONS FORMING A MARKOV SYSTEM

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A **Chebyshev system**, or T**-system**, is a finite system of real functions $\Phi_0(x), \Phi_1(x), \dots, \Phi_n(x)$ $(a < x < b)$, for which the determinant

$$D\begin{pmatrix} \Phi_0, \Phi_1, \dots, \Phi_n \\ x_0,\ x_1, \dots, x_n \end{pmatrix} = \begin{vmatrix} \Phi_0(x_0) & \Phi_0(x_1) & \cdots & \Phi_0(x_n) \\ \Phi_1(x_0) & \Phi_1(x_1) & \cdots & \Phi_1(x_n) \\ \cdots & \cdots & \cdots & \cdots \\ \Phi_n(x_0) & \Phi_n(x_1) & \cdots & \Phi_n(x_n) \end{vmatrix} > 0 \qquad \text{(T)}$$

for all $x_0 < x_1 < \dots < x_n$ from (a, b) (see, for example, ([1])).

A **Markov system**, or M**-system**, is a finite or infinite system of functions $\varphi_0(x), \varphi_1(x), \dots, \varphi_k(x), \dots$, for which all determinants

$$D\begin{pmatrix} \varphi_0, \varphi_1, \dots, \varphi_k \\ x_0,\ x_1, \dots, x_k \end{pmatrix} > 0 \quad \text{for } x_0 < x_1 < \dots < x_k \text{ from } (a, b) \quad (k = 0, 1, 2, \dots). \qquad \text{(M)}$$

It is known that for every Chebyshev system $\Phi_0(x), \Phi_1(x), \dots, \Phi_n(x)$, consisting of continuous functions (only such T-systems will be considered by us), there exists an invertible linear transformation

$$\varphi_k(x) = \sum_{j=0}^{n} c_{kj} \Phi_j(x) \quad (k = 0, 1, \dots, n), \qquad (1)$$

which transforms it into a Markov system*.

Many works are devoted to the study of Chebyshev and Markov systems; however, some important and naturally arising questions, as far as we know, have not yet been clarified. In particular, let us note the following.

A. Is it possible to extend a given finite Markov (Chebyshev) system? If it is possible, then what is the general form of the function $\varphi_{n+1}(x)$?

B. Simple examples show that the functions forming a Markov (Chebyshev) system on the interval (a, b), generally speaking, cannot be extended to a wider interval while preserving property (M) (respectively (T)). Under what conditions is such an extension possible, and what is its general form?

In the present note, as the main result, a representation is given for functions forming a Markov (Chebyshev) system in the form of repeated Stieltjes integrals. From this representation one immediately obtains a complete, and moreover constructive, solution of questions A and B.

Let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_k(x), \ldots$ be a Markov system. For $k = 0$, (M) means that $\varphi_0(x) > 0$. It is clear that the functions
$1,\ \psi_1(x) = \varphi_1(x)/\varphi_0(x),\ \ldots,\ \psi_k(x) = \varphi_k(x)/\varphi_0(x),\ \ldots$ are also a Markov system. Conversely, if

* This, apparently, was first noted by M. G. Krein in a report at the Second All-Union Mathematical Congress.

$1, \psi_1(x), \ldots, \psi_k(x), \ldots$ is a Markov sequence, $\varphi_0(x) > 0$, then $\varphi_0(x), \varphi_0(x)\psi_1(x), \ldots, \varphi_0(x)\psi_k(x), \ldots$ is also a Markov sequence. Therefore one may restrict oneself to considering sequences of the form $1, \psi_1(x), \ldots, \psi_k(x), \ldots$.

For $k = 1$

$$D\begin{pmatrix} 1, \psi_1 \\ x_0, x_1 \end{pmatrix} = \begin{vmatrix} 1 & 1 \\ \psi_1(x_0) & \psi_1(x_1) \end{vmatrix} = \psi_1(x_1) - \psi_1(x_0) > 0, \quad \text{if } x_0 < x_1,$$

i.e. $\psi_1(x)$ is a monotonically increasing function. Therefore, for every $x \in (a, b)$ there exist $\psi_1(x - 0) = \lim_{h\uparrow 0} \psi_1(x + h)$; $\psi_1(x + 0) = \lim_{h\downarrow 0} \psi_1(x + h)$, and $\psi_1(x - 0) = \psi_1(x + 0)$ everywhere except at no more than a countable set of points x.

Theorem 1. *Let $1, \psi_1(x), \ldots, \psi_k(x), \ldots$ $(a < x < b)$ be a Markov sequence. For every $x \in (a, b)$ there exist limiting values $\psi_k(x - 0)$, $\psi_k(x + 0)$ $(k = 1, 2, \ldots)$, and at every point where $\psi_1(x - 0) = \psi_1(x)$ or $\psi_1(x + 0) = \psi_1(x)$, necessarily $\psi_k(x - 0) = \psi_k(x)$ or, respectively, $\psi_k(x + 0) = \psi_k(x)$ $(k = 2, 3, \ldots)$. If at some point of discontinuity of $\psi_1(x)$ one redefines all $\psi_k(x)$, putting $\psi_k(x) = \psi_k(x-0)$ $(k = 1, 2, \ldots)$ or $\psi_k(x) = \psi_k(x+0)$ $(k = 1, 2, \ldots)$, then property (M) is preserved.*

Of course, the system $1, \psi_1(x), \ldots, \psi_k(x), \ldots$ will remain a Markov sequence if the indicated redefinition is carried out on an arbitrary set of discontinuity points of $\psi_1(x)$, in particular at all such points.

In what follows it will be convenient to assume that all functions $\psi_k(x)$ are everywhere in (a, b) continuous from the right: $\psi_k(x + 0) = \psi_k(x)$ $(k = 1, 2, \ldots)$.

Theorem 2. *If $1, \psi_1(x), \ldots, \psi_k(x), \ldots$ $(a < x < b)$ are right-continuous functions forming a Markov sequence, then there exist right-hand relative derivatives*

$$\psi_{k-1}^{(1)}(x) = \lim_{h\downarrow 0} \frac{\psi_k(x + h) - \psi_k(x)}{\psi_1(x + h) - \psi_1(x)} \quad (k = 1, 2, \ldots),$$

where the functions $1, \psi_1^{(1)}(x), \psi_2^{(1)}(x), \ldots$ *are also right-continuous and form in* (a, b) *a Markov sequence.*

The theorem admits the obvious converse: if $1, \psi_1^{(1)}(x), \psi_2^{(1)}(x), \ldots$ is a Markov sequence, and $\psi_1(x)$ is an arbitrary increasing right-continuous function, then $1, \psi_1(x), \int_\alpha^x \psi_1^{(1)}\, d\psi_1, \int_\alpha^x \psi_2^{(1)}\, d\psi_1, \ldots$ is also a Markov sequence ($\alpha \in (a, b)$ is an arbitrary point).

The proofs of Theorems 1 and 2 cannot be given here for lack of space. An important role in them is played by the well-known theorem of M. Fekete on minors of sign-definite matrices (see ([2])), as well as by certain determinant relations given in the monograph of F. R. Gantmakher and M. G. Krein ([3]). In addition, we rely on the following auxiliary proposition, which, in our opinion, is of independent interest.

Let $L = \{x\}$ be a partially ordered set of elements (points on the line, intervals, etc.); let $\varphi_0(x), \varphi_1(x), \ldots, \varphi_k(x), \ldots$ be real functions defined on L such that

$$\varphi_0(x) > 0 \quad \text{on } L;$$

$$D\begin{pmatrix} \varphi_0, \varphi_1, \cdots, \varphi_k \\ x_0, x_1, \ldots, x_k \end{pmatrix} \geq 0 \quad \text{for all } x_0, x_1, \ldots, x_k \in L, \quad x_0 \prec x_1 \prec \cdots \prec x_k$$

$$(k = 1, 2, \ldots).$$

Suppose, further, that there exists a subset $L' \subset L$ such that:

1) for $x', x'' \in L$, $x' \prec x''$, there is always an $x^0 \in L'$ situated "between" them: $x' \prec x^0 \prec x''$.

2)
$$D\begin{pmatrix} \varphi_0, \varphi_1, \cdots, \varphi_k \\ x_0, x_1, \ldots, x_k \end{pmatrix} > 0$$

for all $x_0, x_1, \ldots, x_k \in L'$, $x_0 \prec x_1 \prec \cdots \prec x_k$ $(k = 1, 2, \ldots)$.

Then the latter relation holds for all $x_0 \prec x_1 \prec \cdots \prec x_k$ from L $(k = 1, 2, \ldots)$.

It follows at once from Theorem 2 that the functions $1, \psi_1(x), \ldots, \psi_k(x), \ldots$ forming a Markov sequence can be repeatedly relatively differentiated—each time with respect to the monotone function immediately following the unit. Thus, from the initially given sequence—for uniformity of notation let us write it in the form $1, \psi_1^{(0)}, \psi_2^{(0)}, \ldots$—we obtain Markov sequences

$$\begin{array}{l} 1,\ \psi_1^{(1)},\ \psi_2^{(1)}, \ldots \\ 1,\ \psi_1^{(2)},\ \psi_2^{(2)}, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ 1,\ \psi_1^{(j)},\ \psi_2^{(j)}, \ldots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

Here

$$\psi_k^{(j)}(x) = \lim_{h\downarrow 0} \frac{\psi_{k+1}^{(j-1)}(x+h) - \psi_{k+1}^{(j-1)}(x)}{\psi_1^{(j-1)}(x+h) - \psi_1^{(j-1)}(x)} \qquad (k = 1, 2, \dots;\ j = 1, 2, \dots).$$

It is clear that all the functions $\psi_k^{(j)}(x)$, in particular the functions of the first sequence $\psi_k^{(0)}(x)$, are determined (up to linearly entering constants) through $\psi_1^{(1)}(x), \psi_1^{(1)}(x), \dots, \psi_1^{(j)}(x), \dots$.

We arrive at the following main result.

Theorem 3. *Let the system of functions continuous from the right*

$$1, \psi_1(x), \dots, \psi_k(x), \dots \qquad (a < x < b)$$

form a (finite or infinite) Markov sequence. There exists a (finite or infinite) system of right-continuous monotone increasing functions

$$\sigma_1(x), \sigma_2(x), \dots, \sigma_k(x), \dots$$

and a triangular linear transformation

$$\begin{aligned} &\chi_0(x) = 1, \\ &\chi_1(x) = \psi_1(x) - c_{10}, \\ &\chi_2(x) = \psi_2(x) - c_{21}\psi_1(x) - c_{20}, \\ &\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\chi_k(x) = \psi_k(x) - c_{k,k-1}\psi_{k-1}(x) - \dots - c_{k1}\psi_1(x) - c_{k0} \\ &\qquad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{2}$$

such that

$$\begin{gathered} \chi_0(x) = 1, \\ \chi_1(x) = \int_a^x d\sigma_1, \\ \chi_2(x) = \int_a^x d\sigma_1 \int_a^x d\sigma_2 \left(= \int_a^x \left(\int_a^x d\sigma_2 \right) d\sigma_1 \right), \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \chi_k(x) = \int_a^x d\sigma_1 \int_a^x d\sigma_2 \cdots \int_a^x d\sigma_k \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{gathered} \tag{3}$$

Conversely, if $\alpha \in (a, b)$ is an arbitrary point, and $\sigma_1(x), \sigma_2(x), \dots, \sigma_k(x), \dots$ $(a < x < b)$ is an arbitrary system of continuous, right-continuously monotonically increasing functions, then formulas (3) and (2) determine systems of right-continuous functions $1, \chi_1(x), \dots, \chi_k(x), \dots$ and $1, \psi_1(x), \dots, \psi_k(x), \dots$, which are Markov sequences.

Taking into account the relation (1) between Chebyshev and Markov sequences, we conclude that the functions of a Chebyshev sequence are obtained from functions of the form (3) by means of an invertible linear transformation and multiplication by a positive function. Conversely, such a transformation always converts the system (3) into a Chebyshev sequence.

In short, a Chebyshev sequence is a system of functions linearly equivalent to the "canonical" sequence (3), while a Markov sequence is a system "triangularly equivalent" to the sequence (3).

The results given above—in particular, the main Theorem 3—make it possible to discern a number of properties of Chebyshev and Markov systems which, in our opinion, are not entirely trivial. Let us note in particular the following.

A Markov sequence $1, \psi_1(x), \dots, \psi_k(x), \dots$ $(a < x < b)$ always consists of functions having bounded variation on any interior interval (α, β) $(a < \alpha < \beta < b)$. In order that all functions $\psi_k(x)$ have, at a given point, continuous derivatives of order p, $\partial^p \psi_k / \partial x^p$, it is sufficient that $\psi_1(x), \psi_2(x), \dots, \psi_{p+1}(x)$ have this property. If a given system is a Markov sequence on the intervals (a', b') and (a'', b''), which have a common interior point, then it is a Markov sequence on the interval $(a', b') \cup (a'', b'')$. In particular, if a system is a Markov sequence in some neighborhood of every point $x \in (a, b)$, then it is a Markov sequence on (a, b).

Solution of question A. Every finite Markov (Chebyshev) sequence can be extended, and moreover in infinitely many ways. The (general) method of constructing the function $\varphi_{n+1}(x)$ consists in computing, from the given sequence, the system of monotonically increasing functions $\sigma_1(x), \sigma_2(x), \dots, \sigma_n(x)$ $(\psi_1^{(0)}(x), \psi_1^{(1)}(x), \dots, \psi_1^{(n-1)}(x))$; an arbitrary increasing function $\sigma_{n+1}(x)$ is appended to them, and then formulas (3), (2), and (1) are applied.

Solution of question B. In order that a Markov (Chebyshev) sequence given on the interval $[a, b)$ could be continued beyond the point b (beyond the point a), it is necessary and sufficient that all limiting values $\sigma_k(b-0)$ (respectively, $\sigma_k(a+0)$) be finite. To construct such a continuation, one must correspondingly continue the functions $\sigma_k(x)$ while preserving monotonicity and then apply (3), (2), and (1).

Let us further note that every Markov (Chebyshev) sequence can be redefined on arbitrarily small intervals $(a, a+\varepsilon')$ and $(b-\varepsilon'', b)$, after which it becomes indefinitely extendable beyond both endpoints of the interval (a, b).

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Note: Figure translations are in progress. See original paper for figures.