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Z. O. MELNIK

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Abstract

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Z. O. MELNIK

ON THE SOLVABILITY OF GENERAL MIXED PROBLEMS IN A STRAIGHT CYLINDER FOR ANALYTIC HYPERBOLIC INTEGRO-DIFFERENTIAL EQUATIONS

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In the theory of elliptic and parabolic differential equations there are at present methods that make it possible to study the most general boundary-value problems for a broad class of domains. In the theory of mixed problems for hyperbolic equations the situation is worse. Here only mixed problems in the case of two independent variables have been studied rather fully. In the multidimensional case, for equations of higher orders only mixed problems of a special kind have been studied ([1−3]).

In the present note we prove the solvability of the general mixed problem for a hyperbolic multidimensional integro-differential equation of arbitrary order with analytic coefficients. This note is a direct continuation of the work ([4]).

1. Let G be a given domain (not necessarily bounded) with analytic boundary S in the n-dimensional Euclidean space of points $x = (x_1, \dots, x_n)$.

In the cylinder $G_T : G \times \{0 \le t \le T\}$ we consider the integro-differential equation

$$\sum_{|k| \le m} a^0_{(k)}(x,t) \frac{\partial^{|k|} u(x,t)}{\partial t^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}} + \int_0^t \sum_{|k| \le m-1} a^1_{(k)}(x,t,\tau) \frac{\partial^{|k|} u(x,\tau)}{\partial \tau^{k_0} \partial x_1^{k_1} \cdots \partial x_n^{k_n}} \, d\tau = f(x,t) \tag{1}$$

$$((k) = (k_0, k_1, \dots, k_n); \quad |k| = k_0 + k_1 + \cdots + k_n; \quad a^0_{m,0,\dots,0}(x,t) \equiv 1).$$

The coefficients $a^0_{(k)}$, $a^1_{(k)}$ and the free term f are assumed analytic in $\overline{G}_T$ in all their arguments. In addition, it is assumed that in the expansion

$$\sum_{|k|=m} a^{(0)}_{(k)}(x,t) \lambda^{k_0} \xi_1^{k_1} \cdots \xi_n^{k_n} = \prod_{j=1}^{m} [\lambda - \lambda_j(x,t,\xi)]$$

all the functions $\lambda_j(x,t,\xi)$ are real and distinct for all $(x,t) \in \overline{G}_T$ and for all ξ from the unit sphere. We shall assume them renumbered so that

$$\lambda_1(x,t,\xi) < \lambda_2(x,t,\xi) < \dots < \lambda_m(x,t,\xi).$$

Let $\nu(y) = (\nu_1(y), \dots, \nu_n(y))$ denote the unit vector of the inward normal to S at the point y, and decompose S into $m+1$ parts $S_0, S_1, \dots, S_m$ according to the following rule: S_j consists of all those points $y \in S$ at which, among the functions $\lambda_1(y,t,\nu(y)), \dots, \lambda_m(y,t,\nu(y))$, exactly j are negative

$$(0 \le j \le m;\ 0 \le t \le T).$$

Obviously, any of the sets $S_0, \dots, S_m$ may turn out to be empty; on the other hand, each of these sets need not be connected—it may even consist of several separate pieces. Let $S_T^j = S_j \times \{0 \le t \le T\}$. Then for equation (1) we impose the initial conditions

$$\partial^k u/\partial t^k\big|_{t=0} = g_k(x) \qquad (0 \le k \le m-1,\ x \in \overline{G}) \tag{2}$$

and the boundary conditions

$$\sum_{|k| \le m-1} \left\{ b_{(k)}^{0,j}(y,t)\,\frac{\partial^{|k|} u(y,t)}{\partial t^{k_0}\partial y_1^{k_1}\cdots\partial y_n^{k_n}} + \int_0^t b_{(k)}^{1,j}(y,t,\tau)\,\frac{\partial^{|k|} u(y,\tau)}{\partial \tau^{k_0}\partial y_1^{k_1}\cdots\partial y_n^{k_n}}\,d\tau \right\} = h_j(y,t) \tag{3}$$

$$\left((y,t) \in \bigcup_{k=j}^{m} S_T^k; \quad 1 \le j \le m \right).$$

On S_T^0 no boundary conditions are prescribed.

The initial functions (2), the coefficients, and the free terms of the boundary conditions (3) are assumed to be analytic in all their arguments. In addition, the compatibility conditions of the initial and boundary conditions on the surface S are assumed to be fulfilled:

$$\sum_{|k| \le m-1} b_{(k)}^{0,j}(y,0)\,\frac{\partial^{k_1+\dots+k_n} g_{k_0}(y)}{\partial y_1^{k_1}\cdots\partial y_n^{k_n}} = h_j(y,0) \qquad \left(y \in \bigcup_{k=j}^{m} S_k;\ 1 \le j \le m \right).$$

To these conditions one must add a number of analogous conditions ensuring the continuous differentiability of the solution up to some order r when passing across characteristics issuing from points of S and entering into G_T. Taken together, we shall call these conditions the compatibility conditions of order r.

Introduce the notation:

$$\gamma_{qp}(y,t) = \sum_{s=0}^{m-1} \sum_{k_1+\cdots+k_n=s} b^{0,q}_{m-s-1,k_1,\ldots,k_n}(y,t)\,\nu_1^{k_1}(y)\cdots\nu_n^{k_n}(y)\,\lambda_p^{m-s-1}(y,t,\nu(y))$$

$$\left((y,t)\in\bigcup_{k=q}^{m} S_T^k;\ 1\le q\le m;\ 1\le p\le q\right).$$

It is assumed that at every point $(y,t)\in S_T^j$ the inequality

$$D_j(y,t) = \det\left\|\gamma_{qp}(y,t)\right\|_{q,p=1}^{j} \neq 0 \qquad (1\le j\le m) \tag{4}$$

holds.

Conditions (4) are an analogue of the well-known regularity condition of Ya. B. Lopatinskii in the theory of elliptic boundary-value problems ([5]). It is easy to see that if G is the half-space $x_n > 0$, then conditions (4) coincide with the corresponding conditions from ([2]).

Theorem. *Let the coefficients and free terms of equation (1) and of the boundary conditions (3), and the initial functions (2), be analytic functions of their arguments in their domains of definition, and let the inequalities (4) and the compatibility conditions of order r ($r\ge 0$) for the initial and boundary conditions at the points of the surface S be fulfilled. Then problem (1)–(3) has a piecewise-analytic solution in G_T. The analyticity of the solution is violated only on characteristics issuing from points of the surface S and entering into G_T; on the indicated characteristics the solution belongs to the class C^r.*

The method used to prove the theorem is an analogue of the method of half-space potentials applied in (5) to elliptic boundary-value problems. The application of this method makes it possible to consider mixed problems with boundary conditions of general form. In doing so, the number of conditions that must be prescribed at each point of the lateral surface S_T of the cylinder G_T is immediately apparent. Indeed, suppose that in (1) all coefficients $a^0_{(k)}$ and $a^1_{(k)}$ ($|k| < m$) are identically equal to zero, while the coefficients $a^0_{(k)}$ ($|k| = m$) depend only on t; likewise, for $|k| < m-1$ let $b^j_{(k)} = b^{0,j}_{(k)} = 0$, and let $b^{0,j}_{(k)}, b^{1,j}_{(k)}$ for $|k| = m-1$ be independent of y. Under these conditions, let us consider the auxiliary problem of finding a solution of equation (1) in the half-layer $(x-\xi,\nu(\xi)) > 0$, ($\xi\in S_j$), $0\le t\le T$. From the variables x, by means of an orthogonal linear transformation, we pass to variables y in such a way that the axis Oy_n coincides with the direction $\nu(\xi)$. In the new variables we shall have a problem in the half-space $y_n > 0$. Applying the Fourier transform with respect to $y' = (y_1, \dots, y_{n-1})$, for the transformed function $v(y_n, t)$ we obtain the equation

$$\sum_{i=0}^{m} \left(\sum_{k_1+\dots+k_n=i} a^0_{m-i,k_1,\dots,k_n}(t)\nu_1^{k_1}(\xi)\cdots\nu_n^{k_n}(\xi) \right) \frac{\partial^m v}{\partial t^{m-i}\partial y_n^i} + \dots = f_1(y_n, t).$$

The dots replace terms containing derivatives of v of order $m-1$ and lower. Thus, for v we have obtained a hyperbolic equation of order m with two independent variables. According to (4) and according to the definition of S_j, it is clear that in order for the function v to be determined uniquely, at the point $\xi \in S_j$ it is necessary to prescribe exactly j boundary conditions. From the solution v of the indicated auxiliary problem, the solution of the original problem is constructed without particular difficulties. We note only that, in contrast to the elliptic case, here we arrive not at integral equations for unknown densities, but at integro-differential equations of Volterra type. The proof of the existence of a solution of the resulting system of integro-differential equations has been carried out only under the assumption of analyticity of the data of the original problem. The question of the existence of a solution of problem (1)–(3) in the nonanalytic case remains open for the time being.

2. In the particular case when the boundary conditions have the form

$$\frac{\partial^r u}{\partial \nu^r} = h_r(y,t) \qquad \left((y,t) \in \bigcup_{k=r+1}^{m} S_T^k;\ 0 \le r \le m-1 \right), \tag{5}$$

conditions (4) are always satisfied. Indeed, differentiating (5) $m-r-1$ times with respect to t and noting that

$$\frac{\partial}{\partial \nu} = \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i},$$

we rewrite these conditions in the form:

$$\sum_{r_1+\dots+r_n=r} \frac{r!}{r_1!\cdots r_n!}\nu_1^{r_1}\cdots\nu_n^{r_n} \frac{\partial^{m-1}u(y,t)}{\partial t^{m-r-1}\partial y_1^{r_1}\cdots\partial y_n^{r_n}} = \frac{\partial^{m-r-1}h_r(y,t)}{\partial t^{m-r-1}} \qquad \left((y,t) \in \bigcup_{k=r+1}^{m} S_T^k;\ 0 \le r \le m-1 \right),$$

and, as is not difficult to compute, we shall have

$$D_j(y,t) = (-1)^{j(j+1)/2} \prod_{i=0}^{j-1} \left\{ \lambda_{i+1}^{m-j}(y,t,\nu(y)) \sum_{i_1+\dots+i_n=i} \frac{i!}{i_1!\cdots i_n!}\nu_1^{2i_1}(y)\cdots\nu_n^{2i_n}(y) \right\} \times$$

$$\times \prod_{1\le r<s\le j} [\lambda_s(y,t,\nu(y)) - \lambda_r(y,t,\nu(y))] \qquad \left((y,t) \in S_T^j\right).$$

Since for any $i = 0, 1, \dots, j-1$

$$\sum_{i_1+\dots+i_n=i} \frac{i!}{i_1!\cdots i_n!}\, \nu_1^{2i_1}(y)\cdots\nu_n^{2i_n}(y) \neq 0,$$

then, by virtue of the assumption concerning the characteristic roots $\lambda_r(x,t,\xi)$, we immediately note that $D_j(y,t) \neq 0$ for all $(y,t) \in S_T^j$. In exactly the same way we ascertain that $D_j(y,t) \neq 0$ on every S_T^j, if on $\bigcup_{k=r}^m S_T^k$ any r of the quantities $u, \partial u/\partial\nu, \dots, \partial^{m-1}u/\partial\nu^{m-1}$ are prescribed.

Lviv State University
named after Ivan Franko

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Note: Figure translations are in progress. See original paper for figures.