Soviet-era science, translated into English

G. N. BLOKHINA

1°.** In this note we consider the uniformly elliptic linear equation of second order

1965

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196501.29109

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

G. N. BLOKHINA

THEOREMS OF PHRAGMÉN-LINDELÖF TYPE FOR A LINEAR ELLIPTIC EQUATION OF SECOND ORDER

(Presented by Academician I. G. Petrovskii, 28 I 1965)

1°. In this note we consider the uniformly elliptic linear equation of second order

$$Lu \equiv \sum_{i,k=1}^{m} a_{ik} \frac{\partial^2 u}{\partial x_i \partial x_k} + \sum_{j=1}^{m} b_j \frac{\partial u}{\partial x_j} + cu = 0. \tag{*}$$

Under various assumptions on the coefficients, the nature of the growth is studied for its solution, defined in an unbounded domain and tending to zero on the boundary of the domain. In other words, uniqueness classes are studied for the solution of the Dirichlet problem in unbounded domains. Numerous works have been devoted to theorems of Phragmén-Lindelöf type for equations of elliptic type, including equations of high order and systems, but for domains of a special form ($^{2-7}$). E. M. Landis (^{2}) obtained estimates for the growth of a solution of equation $(*)$, defined in an unbounded domain and tending to zero on the boundary, in terms of volume.

In the present work estimates are obtained in terms of capacity (^{1}). Thanks to this, it is possible to obtain estimates considerably better than in (^{2}). In (^{2}) the growth of the solution is estimated as a function of the smallness of the ratio of the volume of the part of the domain falling inside a ball to the volume of the ball. If this ratio is small, then the estimate obtained in (^{2}) differs little from the estimate obtained in terms of capacity. But if this ratio approaches unity, then the methods of (^{2}) do not allow one to obtain any estimate. This happens because what is essential is not the ratio of the volumes, but the ratio of the capacity of the complement of the part of the domain falling into the ball, relative to the ball, to the capacity of the ball. If the ratio of the volumes is small, then the complement has large volume and, consequently, large capacity. But, of course, the capacity of the complement may be large without the volume of the complement being large, i.e., when the volume of the original domain is small.

We introduce the following notation: D is a domain of m-dimensional $(m > 2)$ space; CE is the complement of the set E to the whole space; Γ_E is the boundary of the set E; $r_n = 2^n$; S_n is the sphere with center at the origin of radius r_n; V_n is the ball with center at the origin of radius r_n; R_n is the spherical layer between the spheres S_n and S_{n+1}; $\overline{E}$ is the closure of the set E; $D_n = \overline{D} \cap R_n$; $D'_n = \overline{CD} \cap \overline{R}_n$; c_n is the capacity of the set D_n; c'_n is the capacity of D'_n; $\alpha_n = c'_n / r_n^{m-2}$; $\beta_n = c_n / r_n^{m-2}$.

2°. Let $a_{ik}(P)$ be continuous together with their first derivatives in D, $a_{ik}(P) \to a_{ik}^0$ as $|P| \to \infty$, $\dfrac{\partial}{\partial x_j} a_{ik}(P) \to 0$ as $|P| \to \infty$, and, moreover,

$$a_{ik}(P) - a_{ik}^0 = O(1/|P|^\alpha), \quad \frac{\partial}{\partial x_j} a_{ik}(P) = O(1/|P|^\alpha).$$

Let $b_j(P) \in C^{(1)}$ in D, $\dfrac{\partial}{\partial x_k} b_j(P)$ and $b_j(P) = O(1/|P|^{1+\alpha})$, and let $c(P) \leq 0$ be continuous and bounded in D. Such an equation will be denoted by $(*')$. Let the domain D satisfy

conditions: a) CD is an unbounded set; b) D_n and D'_n have a regular boundary. We shall call such a domain a **domain of type I**. Then for equation $(*)$ the following theorems are valid.

Theorem 1. Let the domain D of type I be such that $c'_n / r_n^{m-2} \to 0$ as $n \to \infty$. Let $u(P)$ be a solution of equation $(*)$, defined in D and such that $u(P)|_{\Gamma_D} = 0$. Then either $u(P) \equiv 0$ in D, or there exist constants l and N, depending on the equation and the domain, such that for $|P| \geq R > 2^N$

$$\max_{|P| \geq R} |u(P)| \geq M e^{c_1 f[\ln R]},$$

where

$$f(k) = \sum_{n=N}^{kl} \frac{c'_{ln+N}}{r_{ln+N}^{m-2}}, \qquad M = \max_{S_N \cap D} |u(P)|, \quad c_1 \text{ is an absolute constant.}$$

Theorem 2. Let the domain D of type I be such that $c'_n > \beta r_n^{m-2}$ $(0 < \beta < 1)$. Let $u(P)$ be a solution of equation $(*)$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then either $u(P) \equiv 0$ in D, or there exists a constant N, depending on the equation and the domain, such that for $|P| \geq R > 2^N$

$$\max_{|P| \geq R} |u(P)| \geq M R^{\delta_1 \beta},$$

where

$$M = \max_{S_N \cap D} |u(P)|, \qquad \delta_1 \text{ is an absolute constant.}$$

Theorem 3. Let the domain D of type I be such that $c_n < \alpha r_n^{m-2}$ $(0 < \alpha < 1)$. Let $u(P)$ be a solution of equation $(*)$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then either $u(P) \equiv 0$ in D, or there exists a constant N, depending on the equation and the domain, such that for $|P| \geq R > 2^N$

$$\max_{|P| \geq R} |u(P)| \geq M R^{\delta_2 \ln(1/\alpha)},$$

where

$$M = \max_{S_N \cap D} |u(P)|, \qquad \delta_2 \text{ is an absolute constant.}$$

3°. Analogous theorems can also be obtained under other conditions imposed on the coefficients $a_{ik}(P)$ and $b_j(P)$, namely, if $a_{ik}(P)$ satisfy the Hölder condition in D, $a_{ik}(P) \to a_{ik}^0$ as $|P| \to \infty$, and $a_{ik}(P) - a_{ik}^0 = o(1/|P|^{2m-2})$, while $b_j(P), C^{(0,\lambda)}$ in D and $b_j(P) = o(1/|P|^{2m-2})$. In this case we denote the equation by $(*)'$. Let the domain D satisfy the following conditions: a) Γ_D belongs to the class $A^{(1)}$ (for the notation see (9)); b) there exists a ρ_0 such that at each point $Q \in \Gamma_D$ one can construct two balls of radius ρ_0, tangent to Γ_D at the point Q, and such that one of them belongs entirely to the domain D, and the other to its complement CD; c) there exist sets $D_{n,1/n}$ and $D'_{n,1/n}$ such that $D_{n,1/n} \cap R_{n,1/n} = D \cap R_{n,1/n}$ and $D'_{n,1/n} \cap R_{n,1/n} = CD \cap R_{n,1/n}$, where $R_{n,1/n}$ is the spherical layer enclosed between the spheres of radii $2^{n+1/n}$ and $2^{n+1-1/n}$ (one may assume that $\rho_0 \leq 1$); moreover, for $D_{n,1/n}$ and $D'_{n,1/n}$ conditions a) and b) are satisfied if ρ_0 is replaced by $\rho_1 \leq \rho_0 \leq 1$. We shall call such a domain a **domain of type II**. A domain of type II is, for example, a domain for which conditions a) and b) are fulfilled and the spheres S_n intersect Γ_D in such a way that the angle between the tangent planes drawn to the sphere S_n and to Γ_D at the points of intersection of S_n with C_D is bounded below by some quantity $\alpha > 0$. In this case, for $D_{n,1/n}$ and $D'_{n,1/n}$ one may take the set-theoretic sum of all balls of radius ρ_0 contained entirely in D_n or D'_n, respectively.

We note that in all the theorems considered it would have been possible to impose the corresponding conditions not on all D_n or D'_n. It is sufficient that the required conditions be fulfilled for an infinite sequence of sets D_{n_i} or D'_{n_j}. However, in this case the estimates will depend not

only on the capacity c_n or c'_n, and also on the form of the sequence $\{n_i\}$, $\{n_j\}$, respectively.

4°. For the Laplace equation and a domain D of type I, as well as for equation $(*)$ and a domain of type II, one can indicate necessary and sufficient conditions for uniqueness of the solution of the Dirichlet problem in the class of bounded functions.

Theorem 4. *In order that in an unbounded domain D of type I there exist a unique bounded solution of the Dirichlet problem for the Laplace equation, it is necessary and sufficient that the series*

$$\sum_{i=1}^{\infty} \alpha_i$$

diverge,

$$\alpha_i = c_i'/r_i^{m-2}.$$

Theorem 5. *In order that in an unbounded domain D of type II there exist a unique bounded solution of equation $(*)$, it is necessary and sufficient that the series*

$$\sum_{i=1}^{\infty} \alpha_i, \qquad \alpha_i = c(D_{i,1/i'})/r_i^{m-2}.$$

diverge.

It is interesting to note that a domain of type II for which there is no uniqueness of the solution of the Dirichlet problem in the class of bounded functions is such that its complement has no unbounded connected component.

Theorem 6. *Let the domain D of type II be such that there exists a connected component of the complement CD of the domain D which is also an unbounded set. Let $u(P)$ be a solution of the Dirichlet problem for equation $(*'')$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then either $u(P) \equiv 0$ in D, or there exist constants N and β, depending on the equation and the domain, such that for $|P| \geq R > 2^N$*

$$\max_{|P| \geq R} |u(P)| \geq M(\ln R)^{\beta},$$

where

$$M = \max_{S_N \cap D} |u(P)|.$$

5°. Let $a_{ik}(P)$ and $b_j(P)$ satisfy the Hölder condition and be bounded in D, and let $c(P)$ be continuous and bounded in D. In this case we denote the equation by $(*''')$. Suppose that the domain D satisfies the following conditions: a) there exist constant numbers ρ and α such that, for an arbitrary point $Q \in D$, the capacity of the set $D \cap V_{2\rho}^{Q}$, where $V_{2\rho}^{Q}$ is the ball of radius 2ρ with center at Q, is less than $\gamma\rho$ $(0 < \gamma < 1)$; b) let S^n be spheres of radius $n\rho$ with center at the origin of coordinates; then, for any point $Q \in D \cap S^n$, the set $\overline{D} \cap R_n^Q$ has a regular boundary, R_n^Q being the spherical layer contained between the spheres of radii 2ρ and ρ with center at the point Q. We shall call such a domain a domain of cylinder type. In this case the following theorem is valid.

SovietRxiv

Theorem 7. *Let D be a domain of cylinder type. Let $u(P)$ be a solution of the Dirichlet problem for equation $(*''')$ such that $u(P)|_{\Gamma_D} = 0$. Then either $u(P) \equiv 0$ in D, or*

$$\max_{|P| \geq R} |u(P)| \geq M e^{\delta_3 R},$$

where

$$M = \max_{S_0 \cap D} |u(P)|, \qquad \delta_3 = \frac{1}{\rho} \ln \left(\frac{1}{2\alpha} \right).$$

6°. Let $a_{ik}(P)$ and $b_j(P)$ satisfy the Hölder condition and be bounded in D, and let $c(P)$ be continuous and bounded in D. Nothing is assumed about the sign of $c(P)$. We shall denote the equation in this case by $(**)$. Then the following theorems are valid.

Theorem 8. *Let D be a closed bounded domain. Let $u(P)$ be a solution of equation $(**)$ such that $u(P)|_{\Gamma_D} = 0$. Then there exist constants ρ_0 and $\alpha < \rho_0$, depending on the equation and the domain, such that if, for an arbitrary point $Q \in D$, the set*

$$T^Q = CD \cap \overline{V}^Q_{\rho_0},$$

where

$V^Q_{\rho_0}$ is the ball of radius ρ_0 with center at the point Q, has a regular boundary and $c(T^Q) > \alpha$, then $u(P) \equiv 0$ in the domain D.

Theorem 9. Let D be an unbounded domain. Let $u(P)$ be a solution of equation $(**)$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then there exist constants ρ_0 and $\alpha < \rho_0$, depending on the equation and the domain, such that, if for an arbitrary point $Q \in D$ the set

$$T^Q = \overline{CD} \cap V^Q_{\rho_0},$$

where $V^Q_{\rho_0}$ is the ball of radius ρ_0 with center at the point Q, has a regular boundary and $c(T^Q) > \alpha$, then either $u(P) \equiv 0$ in the domain D, or

$$\max_{|P| \geq R} |u(P)| \geq M e^{\gamma R},$$

where

$$M = \max_{D \cap V_{\rho_0}} |u(P)|,$$

and γ depends on α.

Theorem 10. Let D be a bounded closed domain. Let $u(P)$ be a solution of $(**)$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then there exist constants ρ_0 and $\beta < \rho_0$, depending on the equation and the domain, such that, if for an arbitrary point $Q \in D$ the set

$$T^Q = \overline{D} \cap R^Q$$

$$(R^Q = V_{PQ} \setminus V^Q_{\rho/2})$$

has a regular boundary and $0 < c(T^Q) < \beta$, then $u(P) \equiv 0$ in D.

Theorem 11. Let D be an unbounded domain. Let $u(P)$ be a solution of equation $(**)$ in the domain D such that $u(P)|_{\Gamma_D} = 0$. Then there exist constants ρ_0 and $\beta < \rho_0$, depending on the equation and the domain, such that, if for an arbitrary point $Q \in D$ the set

$$T^Q = \overline{D} \cap R^Q$$

$$(R^Q = V_{PQ} \setminus V^Q_{\rho/2})$$

has a regular boundary and $0 < c(T^Q) < \beta$, then either $u(P) \equiv 0$ in D, or

$$\max_{|P| \geq R} |u(P)| \geq M e^{\delta R},$$

$$M = \max_{D \cap V^0_{\rho/2}} |u(P)|,$$

and δ depends on β.

In conclusion the author expresses deep gratitude to E. M. Landis for his guidance and constant attention.

Moscow State University
named after M. V. Lomonosov

Received
25 I 1965

CITED LITERATURE

1. M. V. Keldysh, *Uspekhi Mat. Nauk*, vol. 8 (1940).
2. E. M. Landis, *Uspekhi Mat. Nauk*, **14**, no. 1 (1959); **18**, no. 1 (1963).
3. R. Nevanlinna, *Single-Valued Analytic Functions*, Moscow-Leningrad, 1941.
4. P. D. Lax, *Amer. Math. Soc. Translations*, Mathematics, **3**, 4 (1959).
5. M. A. Evgrafov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **27**, no. 4 (1963).
6. D. Hilbert, D. Courant, *Amer. Math. Soc. Translations*, Mathematics, **2**, 6 (1958).
7. D. Courant, *Amer. Math. Soc. Translations*, Mathematics, **2**, 6 (1958).

8. Ya. Ya. Stokalo, *Investigation of the Qualitative Theory of Differential Equations with Partial Derivatives*, Lvov, 1961.
9. C. Miranda, *Equations with Partial Derivatives of Elliptic Type*, 1957.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.