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L. N. SHEVRIN

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Abstract

Full Text

MATHEMATICS

L. N. SHEVRIN

ON LOCALLY FINITE SEMIGROUPS

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The main result of the present note is the following.

Theorem. *A bundle of locally finite semigroups is locally finite.*

As a very special case of this theorem we obtain, for example, the well-known (see ^(1, 2)) proposition on the local finiteness of the direct product of semigroups of idempotents, proved by McLean ⁽³⁾. From the theorem there also follows a stronger proposition, which constitutes the main result of the paper of Grillet and Riesz ⁽⁴⁾ (see Corollary 2 below). At the same time, the nature of the corresponding results is revealed more deeply. We shall also indicate other consequences of the theorem.

Let us recall some definitions. If a semigroup Γ has a decomposition into subsemigroups Γ_α (α runs through some set of indices A), which are congruence classes in Γ , then Γ is called a **bundle** of the semigroups Γ_α . A bundle is called **commutative** if, for any $\alpha, \beta, \delta \in A$, from $\Gamma_\alpha \Gamma_\beta \subseteq \Gamma_\delta$ it follows that $\Gamma_\beta \Gamma_\alpha \subseteq \Gamma_\delta$. A bundle is called **matrix** if its components can be provided with pairs of indices $\Gamma_{\xi\eta}$ ($\xi \in X, \eta \in Y$) in such a way that $\Gamma_{\xi_1\eta_1} \Gamma_{\xi_2\eta_2} \subseteq \Gamma_{\xi_1\eta_2}$ for any $\xi_1, \xi_2 \in X, \eta_1, \eta_2 \in Y$. We shall call a bundle **left (right)** if all its components are left (right) ideals.

The factor semigroup whose elements are the components of a given commutative bundle Γ is a commutative semigroup of idempotents and, consequently, is locally finite; therefore any finite number of elements of Γ is contained in a subsemigroup which is a bundle of a finite number of components of the original commutative bundle. In other words, the following is true.

Lemma 1. *A commutative bundle of semigroups has a local system of subsemigroups, each of which is the union of a finite number of components of this bundle.*

The concept of a local system of subsemigroups is used in the usual sense, analogous to that in group theory (see ⁽⁵⁾).

Lemma 2. *A commutative bundle of a finite number of semigroups has a finite series of ideals, the first term of which is one of the components of the bundle, and the remaining factors are isomorphic to semigroups obtained from other components by adjoining zero externally.*

Proof. We pass to the factor semigroup corresponding to the given bundle, which, as is known, is a (lower) semilattice with respect to the following relation \leq :

$$x \leq y \leftrightarrow xy = x.$$

Let P be the indicated finite semilattice. It obviously has a zero (least element). Arrange all the elements of P in nondecreasing order of their heights (see ⁽⁶⁾): x_1, \dots, x_m . It is clear that if the meet of distinct elements $x, y \in P$ is different from each of these elements, then it has height less than both the height of x and the height of y ; if it is equal to one of them, then the height of this element is less than the height of the other. Hence it follows that the set of elements x_1, \dots, x_i will be an ideal for every $i = 1, \dots, m$. Passing now to the corresponding components of the bundle X_1, \dots, X_m , we construct the series of ideals

$$X_1 \subset X_1 \cup X_2 \subset \dots \subset X_1 \cup \dots \cup X_m.$$

The lemma is proved.

Let θ be an abstract theoretical-semigroup property. A semigroup possessing the property θ will, as usual, be called a θ -semigroup. By $L\theta$ we shall denote the property of a semigroup to possess a local system of subsemigroups that are θ -semigroups. Consider the following conditions that the property θ may satisfy.

- A. The semigroup obtained from a θ -semigroup by external adjunction of a zero is a θ -semigroup.
- B. An ideal extension of a θ -semigroup by means of a θ -semigroup is a θ -semigroup.
- C. A semigroup anti-isomorphic to a θ -semigroup is a θ -semigroup.
- D. The left sum of θ -semigroups is a θ -semigroup.
- E. θ is a local property, i.e. $L\theta = \theta$.

From Lemmas 1 and 2 the following follows.

Proposition 1. *If the property θ satisfies conditions A and B, then the commutative sum of θ -semigroups is an $L\theta$ -semigroup.*

Proposition 2. *If the property θ satisfies conditions C and D, then the matrix sum of θ -semigroups is a θ -semigroup.*

Proof. The matrix sum of semigroups of a given family is the left (right) sum of the right (left) sums of certain semigroups of this family. Indeed, using the notation from the definition of a matrix sum, we conclude that the semigroups $\Gamma_{\xi\eta}$ with fixed second index form a right sum, and the unions

$$H_\eta = \bigcup_{\xi \in X} \Gamma_{\xi\eta}$$

are the components of the decomposition of the original semigroup into a left sum.

A semigroup anti-isomorphic to a right sum of semigroups is the left sum of semigroups anti-isomorphic to the original ones; therefore, for a property θ satisfying conditions C and D, we have: the right sum of θ -semigroups is a θ -semigroup. Hence, from the assertion made at the beginning of the preceding paragraph, we obtain the required result.

Since an arbitrary sum of semigroups of a given family, according to Clifford' s theorem ((⁷), see also (^{1,2})), is a commutative sum of the matrix sums of certain semigroups of this family, the following follows directly from Propositions 1 and 2.

Proposition 3. *If the property θ satisfies conditions A-E, then the sum of θ -semigroups is a θ -semigroup.*

We now pass to the case of interest to us, when θ is local finiteness. The proof of the following lemma is carried out analogously to the proof of item 3 of Theorem 2.7 of the work (⁸).

Lemma 3. *An ideal extension of a locally finite semigroup by means of a locally finite semigroup is itself locally finite.*

Lemma 4. *The left sum of locally finite semigroups is locally finite.*

The proof of Lemma 4 is based on the following auxiliary lemma.

Lemma 5. *Let elements a_1, \dots, a_n of a given semigroup generate an infinite subsemigroup. Then there exists an infinite sequence x_1, \dots, x_k, \dots , each term of which is some element among a_1, \dots, a_n , such that for any natural k, l, p, q the following condition holds:*

$$(*) \text{ if } l \neq q, \text{ then } x_k x_{k+1} \dots x_{k+l} \neq x_p x_{p+1} \dots x_{p+q}.$$

Proof. Each element of the subsemigroup $H = \{a_1, \dots, a_n\}$ is represented in the form of some word in the alphabet a_1, \dots, a_n , and among such representations there exist words of minimal length. Fix some such irreducible representations. By virtue of the infinitude of H , at least one of the letters a_1, \dots, a_n occurs infinitely many times

as the first letter in these words. Denote it by x_1 , and denote by H_1 the set of all corresponding words. The set H_1 is infinite; therefore at least one of the letters a_1, \dots, a_n occurs as the second letter in the words of H_1 infinitely many times. Denote it by x_2 . Continuing this process by induction, we obtain a sequence $x_1, x_2, \dots, x_k, \dots$ and, respectively, sets $H_1, H_2, \dots, H_k, \dots$. We shall show that

the indicated sequence is the desired one. Suppose that for some k, l, p, q one has

$$x_k x_{k+1} \dots x_{k+l} = x_p x_{p+1} \dots x_{p+q},$$

while $l \neq q$. Put $s = \max(k + l, p + q)$. The set H_s is infinite; hence it contains words of arbitrarily large length. Take in H_s an arbitrary word h whose length is greater than s . Each of the words $x_k x_{k+1} \dots x_{k+l}$, $x_p x_{p+1} \dots x_{p+q}$ then enters into h as a segment. The lengths of these words are unequal, since $l \neq q$. Replacing in h the longer of the indicated words by the other word, we obtain the same element, represented by a word of smaller length. This contradicts the irreducibility of the chosen representations for the elements of H . The contradiction obtained proves the lemma.

Proof of Lemma 4. Let Γ be a left union of locally finite semigroups Γ_α ($\alpha \in A$), and let a_1, \dots, a_n be arbitrary elements of Γ . We shall show that the subsemigroup $\{a_1, \dots, a_n\}$ is finite. We use induction on n . For $n = 1$ this assertion is trivial, since Γ is a periodic semigroup. Suppose the assertion has been proved for $n - 1$. Assume that the subsemigroup is infinite. Take a certain sequence x_1, x_2, \dots of the kind considered in Lemma 5. At least one of the letters a_1, \dots, a_n occurs in it infinitely many times; without loss of generality we assume that it is a_1 . In the given sequence consider all possible segments of the form $a_1 x_k \dots x_{k+l} a_1$, where $x_i \neq a_1$ for $i = k, \dots, k + l$. From condition () and from the induction hypothesis it follows that, in the segments under consideration, the numbers l are, in the aggregate, bounded by some number l_0 ; whence, in turn, it follows that there are only finitely many distinct words of the indicated form. For each word $a_1 x_k \dots x_{k+l} a_1$ consider the corresponding element $b_k = x_k \dots x_{k+l} a_1$. There are finitely many elements b_k , and all of them belong to one and the same component Γ_α —the one to which a_1 belongs. But from the infinitude of the sequence under consideration and from condition () it follows that the subsemigroup generated by the indicated elements is infinite. This contradicts the local finiteness of the subsemigroup Γ_α . The contradiction obtained proves the lemma.

Since conditions A, B, and D for local finiteness are satisfied trivially, the theorem formulated at the beginning of the note follows directly from Lemmas 3 and 4 and Proposition 3.

Since a semigroup possessing a decomposition into subgroups is, as is well known (see $(7, 1, 2)$), a commutative union of matrix unions of groups, the following is true.

Corollary 1. *A semigroup possessing a decomposition into locally finite groups is locally finite.*

Since a semigroup with the identity relation $x^r = x$ ($r > 1$) possesses, as is easy to see, a decomposition into subgroups, it follows in turn from Corollary 1 that

Corollary 2 (Green and Rees theorem). *The following assertions are equivalent:*

(A_r) An arbitrary semigroup with the identity relation $x^r = x$ is locally finite.

(B_r) An arbitrary group with the identity relation $x^{r-1} = 1$ is locally finite.

Corollary 3 (McLean theorem). *An arbitrary semigroup of idempotents is locally finite.*

Let us indicate several corollaries concerning semigroups with certain types of subsemigroup structures. Taking into account the results of the paper ⁽⁹⁾, semi- by using local finiteness of an arbitrary nilpotent semigroup and applying Lemma 3 and the theorem of the present note, we obtain

Corollary 4. *A semigroup with a Dedekind structure of subsemigroups is locally finite if and only if all of its subsemigroups are locally finite.*

Let us note that examples of periodic groups with a Dedekind structure of subgroups that are not locally finite are still unknown. If no such groups exist, then, by Corollary 4, any semigroup with a Dedekind structure of subsemigroups is locally finite. In any case, the following holds.

Corollary 5. *An arbitrary semigroup with a distributive structure of subsemigroups is locally finite.*

Taking into account the results of the paper [10], we obtain

Corollary 6. *An RK-semigroup is locally finite if and only if all of its maximal subgroups are locally finite.*

In conclusion we note the following question, concerning a possible generalization of the theorem of the present note and suggested by Corollary 1 from it: will an arbitrary semigroup possessing a decomposition into locally finite semigroups be locally finite?

Ural State University
named after A. M. Gorky

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