ON A CLASS OF PARTITIONS OF LOBACHEVSKY SPACE

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Abstract

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The theory of Fedorov groups of the Lobachevsky plane was constructed in Poincaré' s famous memoirs, but until recently there existed only isolated examples, finite in number, of Fedorov groups in Lobachevsky space. In recent years interest in the study of such groups has grown considerably, and numerous examples have been obtained of infinite series of such groups both in Lobachevsky space and in other analogous spaces; all of them, however, were obtained by rather difficult arithmetical methods with the aid of the theory of integral automorphisms of various forms. There is a conjecture of Borel, not refuted to this day, that all such groups are arithmetic ([1,2]) and others. The author of the present note gives a very simple geometric method for constructing an infinite number of nonisomorphic Fedorov groups of three-dimensional Lobachevsky space.

Take in Lobachevsky space some plane and consider its regular partition into (equal) regular n-gons, which, as is known, is possible for any value of n. In the construction considered by us, only those cases are important in which the number m of polygons meeting at a vertex is equal to 3, 4, or 5. In this case $n > 3$.

Suppose that one of such partitions of the plane is given. Erect to this plane perpendiculars at all vertices of these polygons, and lay off on them, on both sides of the plane, equal segments h, such that the regular n-gons whose vertices are the endpoints of the perpendiculars (lying on one side of the plane) form at their vertices polyhedral angles each of which is exactly equal to one of the central angles of regular polyhedra. The infinite convex polyhedron bounded by the two resulting infinite polyhedral surfaces formed by these n-gons shall be called a **lens**. Such a lens naturally decomposes into an infinite number of prisms adjacent by their lateral faces. We shall attach the obtained lenses to one another along whole faces. The totality of all prisms of these lenses will constitute precisely such a polyhedron as was considered by A. D. Aleksandrov in ([3]); moreover, in the star of prisms meeting at any edge, these prisms will not enter one another.

Indeed, if this is a lateral edge, then the star of prisms meeting at it corresponds to the star of n-gons at a vertex of our partition of the plane. If, however, the edge under consideration is an edge of the base of a prism, then the lenses meeting at it do not enter one another, since the star of lenses meeting at this edge corresponds to the star at a vertex of a regular partition of the sphere into regular m-gons. Therefore the prisms meeting at this edge and belonging to different lenses do not enter one another; and the prisms meeting at this edge and belonging to the same lens do not enter one another, since they do not enter one another in the lens. According to Theorems 1 and 2 of A. D. Aleksandrov ([3]), the prisms under consideration form a normal partition of Lobachevsky space. This partition

correct, since the construction algorithm is the same if one starts from any prism in any lens.

In the group of the partition thus obtained there are rotation axes of order n. It is easy to see that all other possible axes of the partition under consideration have order not greater than 5. Therefore, if $n > 5$, then an axis of order n is the highest in the group. On the other hand, as is easy to show, any finite cyclic subgroup of order n of a group of motions of the first kind is realized only as an axis of order n. Therefore, if we carry out two such constructions—one with $n = n_1$ and the other with $n = n_2$, where $n_1 \neq n_2$ and $n_1, n_2 > 5$, we obtain two groups whose highest finite cyclic subgroups are different, and hence the groups themselves are nonisomorphic. The preceding construction gives, if one takes $n = 6, 7, ...$, a countably infinite set of nonisomorphic Fedorov groups in three-dimensional Lobachevsky space. From the construction follows at least the countability of the number of distinct topological types of polyhedra—n-gonal prisms—each of which normally and regularly partitions Lobachevsky space. If, in addition, the prisms are subdivided by planes that bisect the dihedral angles at the lateral edges, then we obtain a countable number of normal and regular partitions into polyhedra of one and the same topological type—a triangular prism.

It is possible to construct examples of partitions of Lobachevsky space whose groups contain several axes of arbitrary orders; from this follows the countability of the number of topological types of polyhedra, each of which partitions Lobachevsky space in a countable number of ways.

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Note: Figure translations are in progress. See original paper for figures.