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## Abstract

## Full Text

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# ON LYAPUNOV THEOREMS FOR HEAT POTENTIALS

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In potential theory, an important role is played by the profound investigations of A. M. Lyapunov ([1]) concerning the properties of various potentials (in particular, their behavior near the boundary) and the related solutions of a number of boundary-value problems for elliptic equations. In our note we present some theorems of Lyapunov type (Theorems 3-5) for heat potentials, establishing those conditions on the density and on the surface under which the heat potential of a simple layer, of a double layer, and the normal derivative of the heat potential of a simple layer possess prescribed smoothness.

As one of the applications of the Lyapunov theorems, we prove the existence of a $(2+\alpha)$-smooth solution of the third boundary-value problem for a linear parabolic equation of the second order. The method of analytic continuation used here is essentially based on the $(2+\alpha)$ a priori estimate (up to the boundary) for solutions of the indicated problem, established by the author jointly with V. N. Maslennikova in ([3]).

§ 1. **Definition.** Let, for a closed surface $\Gamma$ of $(n+1)$-dimensional space $(x,t) = (x_1, x_2, \dots, x_n; t)$, enclosed between the hyperplanes $t = 0$ and $t = T > 0$, there exist a constant $R > 0$ such that in every sphere $S_R(x,t)$ with center at the point $(x,t) \in \Gamma$ and radius $R$ the surface $\Gamma$ admits a representation of the form

$$x_i = g(x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots, x_n; t) \qquad (1)$$

(without loss of generality we shall take $i = n$).

A. If $g(x,t)$ in (1) is such that $\partial^{2m-p} g / \partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}$ $(p = 0, 1, \dots, m)$ satisfies, with respect to $t$, a Hölder condition with exponent $(1+\alpha)/2$, and $\partial^{2m-p+1} g / \partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}$ $(p = 0, 1, \dots, m)$ satisfies, with respect to $x$ and $t$, Hölder conditions with exponents $\alpha_1$ and $\alpha/2$, respectively, then we shall say that the surface $\Gamma$ is of type

$$\mathcal{L}^{m,1,(1+\alpha)/2}_{2m+1,\alpha_1,\alpha/2} \quad (0 < \alpha,\ \alpha_1 \le 1,\ m = 0, 1, 2, \dots).$$

B. If $g(x,t)$ in (1) is such that $\partial^{2m-p}g/\partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}$ satisfies, with respect to $x$ and $t$, Hölder conditions with exponents $\alpha_1$ and $\alpha/2$, respectively, and, for $m \geq 1$ (for $m = 0$ this condition is omitted), $\partial^{2m-p-1}g/\partial t^p \partial x_1^{l_1} \cdots \partial x_{n-1}^{l_{n-1}}$ $(p = 0, 1, \dots, m-1)$ satisfies, with respect to $t$, a Hölder condition with exponent $(1+\alpha)/2$, then we shall say that the surface $\Gamma$ is of type

$$\mathcal{L}_{2m-1,1,(1+\alpha)/2}^{m,\alpha_1,\alpha/2} \quad (0 < \alpha,\ \alpha_1 \leq 1)$$

for $m = 1, 2, \dots$, and of type $\mathcal{L}^{0,\alpha_1,\alpha/2}$ for $m = 0$.

Introduce the notation:

$$G(x-\xi, t-\tau) = (t-\tau)^{-n/2} \exp\left\{-\frac{1}{4}(t-\tau)^{-1}\sum_{i=1}^{n}(x_i-\xi_i)^2\right\};$$

$\Gamma_\tau$ is the section of the surface $\Gamma$ by the hyperplane $t = \tau$ $(0 \leq \tau \leq T < +\infty)$; $n(x,t)$ is the interior normal to the section $\Gamma_t$ at the point $(x,t) \in \Gamma_t$; $Q_T$ is the domain of the space $(x,t)$ bounded by the lateral surface $\Gamma$ and the hyperplanes $t = 0$ and $t = T > 0$.

**Theorem 1.** If the surface $\Gamma$ is of type ${}_{1,\alpha,\alpha/2}^{0,1,(1+\alpha)/2}$ $(0 < \alpha \leqslant 1)$ and the density $\Phi(\xi,\tau)$ is a bounded integrable function, then the heat potential of a simple layer

$$U(x,t) = \int_0^t d\tau \iint_{\Gamma_\tau} G(x-\xi, t-\tau)\Phi(\xi,\tau)\, d\sigma_\xi(\tau)$$

satisfies, in the domain $\{(x,t), |x_i| < +\infty,\ i = 1, 2, \dots, n;\ 0 \leqslant t \leqslant T\}$, the Hölder conditions in $x$ and $t$ with exponents $\alpha'$ and $1/2$, respectively, where $\alpha'$ is any number for which $0 < \alpha' < 1$.

**Theorem 2.** If the surface $\Gamma$ is of type ${}_{1,\alpha,\alpha/2}^{0,1,(1+\alpha)/2}$ and the density $\Phi(\xi,\tau)$ is a bounded integrable function, then the normal derivative of the heat potential of a simple layer $V(x,t) = \partial U(x,t)/\partial n(x,t)$ and the heat potential of a double layer

$$W(x,t) = \int_0^t d\tau \iint_{\Gamma_\tau} \frac{\partial G(x-\xi, t-\tau)}{\partial n(\xi,\tau)}\Phi(\xi,\tau)\, d\sigma_\xi(\tau)$$

satisfy on the surface $\Gamma$ $((x,t) \in \Gamma)$ the Hölder conditions in $x$ and $t$ with exponents $\alpha'$ and $\alpha'/2$, respectively, where $0 < \alpha' < \alpha$, $\alpha'$ arbitrary.

**Remark.** If the density $\Phi(\xi,\tau)$ is continuous on $\Gamma$, then for $V(x,t)$ and $W(x,t)$ the Joukovsky formulas ([2]) hold, determining the jumps of these potentials when passing through the surface $\Gamma$.

**Definition.** Let $f(x_1, \dots, x_k; t)$ be defined in a domain $B$ and satisfy in this domain one of the conditions A and B of the definition of type  for the surface $\Gamma$ (where $n-1=k$). We shall then say that the function $f(x,t)$ belongs in the domain $B$ to the class $H^{m,1,(1+\alpha)/2}_{2m+1,\alpha_1,\alpha/2}(B)$ or $H^{m,\alpha_1,\alpha/2}_{2m-1,1,(1+\alpha)/2}(B)$, respectively.

**Theorem 3.** If the surface $\Gamma$ is of type ${}^{m,\alpha,\alpha/2}_{2m-1,1,(1+\alpha)/2}$, $0<\alpha<1$, $m=1,2,\dots$ (or ${}^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}$, $m=1,2,\dots$) and the density $\Phi(\xi,\tau)$ belongs on the surface $\Gamma$ to the class $H^{m-1,\alpha,\alpha/2}_{2(m-1)-1,1,(1+\alpha)/2}(\Gamma)$ (or $H^{m-1,1,(1+\alpha)/2}_{2(m-1)+1,\alpha,\alpha/2}(\Gamma)$), and moreover

$$\frac{\partial^k}{\partial\tau^k}\Phi(\xi,0)=0, \qquad (\xi,0)\in\Gamma, \qquad 0\leqslant k\leqslant m-1, \tag{2}$$

then the potentials $V(x,t)$ and $W(x,t)$ belong on the surface $\Gamma$ to the class $H^{m-1,1,(1+\alpha)/2}_{2(m-1),\alpha',\alpha'/2}(\Gamma)$ (respectively to the class $H^{m,\alpha',\alpha/2}_{2m-1,1,(1+\alpha)/2}(\Gamma)$), where $0<\alpha'<\alpha$, $\alpha'$ arbitrary.

**Theorem 4.** If the surface $\Gamma$ is of type ${}^{m,\alpha,\alpha/2}_{2m-1,1,(1+\alpha)/2}$, $0<\alpha<1$, $m=1,2,\dots$ (or ${}^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}$, $0<\alpha<1$, $m=1,2,\dots$) and the density $\Phi(\xi,\tau)$ belongs on the surface $\Gamma$ to the class $H^{m-1,\alpha,1/2}_{2(m-1)-1,1,(1+\alpha)/2}(\Gamma)$ (or $H^{m-1,1,(1+\alpha)/2}_{2(m-1)+1,\alpha,\alpha/2}(\Gamma)$), and moreover (2) holds, then the heat potential of a simple layer $U(x,t)$ belongs on the surface $\Gamma$ to the class $H^{m-1,1,(1+\alpha)/2}_{2(m-1)+1,\alpha,\alpha/2}(\Gamma)$ (respectively $H^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}(\Gamma)$).

**Theorem 5.** Let the surface $\Gamma$ be of type ${}^{m,\alpha,\alpha/2}_{2m-1,1,(1+\alpha)/2}$, $0<\alpha<1$, $m=1,2,\dots$ (or ${}^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}$) and let the density $\Phi(\xi,\tau)$ belong on the surface $\Gamma$ to the class $H^{m-1,1,(1+\alpha)/2}_{2(m-1),\alpha,\alpha/2}(\Gamma)$ (or to the class $H^{m,\alpha,\alpha/2}_{2m-1,1,(1+\alpha)/2}(\Gamma)$), and moreover (2) holds. Then the heat potential of a simple layer $U(x,t)$ belongs in the domain $\overline{Q}_T$ to the class $H^{m,\alpha,\alpha/2}_{2m-1,1,(1+\alpha)/2}(\overline{Q}_T)$ (respectively $H^{m,1,(1+\alpha)/2}_{2m+1,\alpha,\alpha/2}(\overline{Q}_T)$).

The proof of Theorems 1-5 (rather complicated, especially for Theorems 3-5) is carried out by classical methods close to those presented in ([1]) for the ordinary theory of potential.

## § 2. Consider the third boundary-value problem for the heat-conduction equation

$$\sum_{i=1}^{n}\frac{\partial^2 u}{\partial x_i^2}-\frac{\partial u}{\partial t}=f(x,t), \qquad (x,t)\in Q_T; \tag{3}$$

$$u(x,0)=\psi(x), \qquad x\in\Omega=\overline{Q}_T\cap\{t=0\}; \tag{4}$$

$$\frac{\partial u(x,t)}{\partial n(x,t)}+b(x,t)u(x,t)=\varphi(x,t), \qquad (x,t)\in\Gamma. \tag{5}$$

**Theorem 6.** Let the surface $\Gamma$ be of type $J^{1,\beta,\beta/2}_{1,1,(1+\beta)/2}$ ($0 \leq \alpha < \beta < 1$, where $\beta$ is arbitrary), let the function $f(x,t)$ belong to the class $H^{0,\alpha,\alpha/2}(\overline{Q}_T)$, let the function $\psi(t)$ have partial derivatives $\partial^2\psi/\partial x_i \partial x_j$ satisfying in $x$ in $\Omega$ a Hölder condition with exponent $\alpha$, and let the functions $b(x,t)$ and $\varphi(x,t)$ belong to the class $H^{0,1,(1+\alpha)/2}_{1,\alpha,\alpha/2}(\Gamma)$, with the right-hand side $f$ of (3), the initial function $\psi$, and the boundary function $\varphi$ being compatible, in view of equation (3), on the edge $\Omega \cap \Gamma$. Then there exists a solution $u(x,t)$ of problem (3)–(5) belonging to the class $H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}(\overline{Q}_T)$.

**Proof.** In view of the conditions imposed on $f$ and $\psi$, and of Gevrey' s results ([2]) for the heat volume potential and the Poisson integral, instead of problem (3)–(5) one may consider the problem $(3^0), (4^0), (5)$, putting in (3) and (4) $f(x,t) \equiv \psi(x) \equiv 0$. The solution of problem $(3^0), (4^0), (5)$ is sought in the form of a heat potential of a simple layer with unknown density $\Phi(\xi,\tau)$. The use of boundary condition (5) leads, in view of the remark to Theorem 2, to a Volterra integral equation of the second kind for determining $\Phi(\xi,\tau)$. Theorems 1–4 show that $\Phi(\xi,\tau)$ then possesses all the smoothness properties required in Theorem 5 (with $m = 1$), the application of which completes the proof of Theorem 6.

## § 3. Consider the third boundary-value problem for the parabolic equation

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + c(x,t)u - \frac{\partial u}{\partial t} = f(x,t), \qquad (x,t) \in Q_T \tag{6}$$

with the initial condition (4) and the boundary condition

$$\frac{\partial u(x,t)}{\partial N(x,t)} + b(x,t)u(x,t) = \varphi(x,t), \qquad (x,t) \in \Gamma, \tag{7}$$

where $\partial/\partial N(x,t)$ is the derivative along the conormal to the surface $\Gamma$ at the point $(x,t)$.

The $(2+\alpha)$ a priori estimate for the solution of the third boundary-value problem (6), (4), (7) in the closed domain $\overline{Q}_T$, obtained by the author jointly with V. N. Maslennikova in ([3]), makes it possible, with the aid of Theorem 6, to prove by the classical method of continuation with respect to a parameter the following existence theorem.

**Theorem 7.** Let the surface $\Gamma$ be of type $J^{1,\beta,\beta/2}_{1,1,(1+\beta)/2}$ ($0 < \alpha < \beta < 1$, where $\beta$ is arbitrary), let equation (6) be of parabolic type in $\overline{Q}_T$, and let the coefficients (6) $a_{ij}(x,t)$, $b_i(x,t)$, and $c(x,t)$ belong to the class $H^{0,\alpha,\alpha/2}(\overline{Q}_T)$. Let the functions $f(x,t)$ (in $\overline{Q}_T$), $\psi(x)$ (in $\Omega$), $b(x,t)$, $\varphi(x,t)$, and also $a_{ij}(x,t)$ (on $\Gamma$) satisfy the conditions of Theorem 6, with $f(x,t)$, $\psi(x)$, and $\varphi(x,t)$ being compatible in view of equation (6). Then there exists a solution $u(x,t)$ of problem (6), (4), (7) belonging to the class $H^{1,\alpha,\alpha/2}_{1,1,(1+\alpha)/2}(\overline{Q}_T)$.

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## CITED LITERATURE

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*Note: Figure translations are in progress. See original paper for figures.*