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Abstract

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ON THE SOLUTION OF AN EQUATION OF INFINITE ORDER IN A UNIQUENESS CLASS

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In the monograph ([1]), P. Sikkema studied an operator of the form

$$L(F) \equiv \sum_{n=0}^{\infty} c_n F^{(n)}(z), \tag{1}$$

where the function $F(z)$ is assumed to be entire, and the sequence $\{c_n\}$ $(n = 0, 1, 2, ...)$ is a given sequence of complex numbers. The operator $L(F)$ is called applicable to an entire function $F(z)$ if the series (1) converges for every finite z.

In ([1]), necessary and sufficient conditions are given for the applicability of the operator to entire functions of the class $[\nu, \tau]$; questions of the growth of the operator $L(F)$ depending on the growth of the function $F(z)$ are studied; and the question of the existence of a solution of the equation $L(F) = h(z)$, where $h(z)$ is a given function, is considered. In the present note, operators of a more general type are considered.

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \neq 0 \quad (n = 0, 1, 2, ...)$$

be an entire function of order ρ and type $\sigma \neq 0, \infty$, satisfying the condition: the limit exists

$$\lim_{n\to\infty} \{n!^{1/\rho} |a_n|\}^{1/n} = (\sigma\rho)^{1/\rho}.$$

With the aid of the function $f(z)$, generalized derivatives are generated, first introduced by A. O. Gel' fond and A. F. Leont' ev in ([2]). Namely, if

$$F(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then

$$D^n F(z) = \sum_{k=0}^{\infty} b_{n+k} \frac{a_k}{a_{n+k}} z^k$$

is the generalized derivative of order n of the function $F(z)$.

Next, let $\{c_n\}$ $(n = 0, 1, 2, ...)$ be a sequence of complex numbers. Consider the operator

$$M(F) \equiv \sum_{n=0}^{\infty} c_n D^n F. \tag{2}$$

We note that if $f(z) = e^z$, then the operator (2) becomes the operator (1).

The following theorem contains conditions for the applicability of the operator $M(F)$.

Theorem 1. *For the applicability of the operator $M(F)$ to entire functions of the class $[\nu, \tau]$, it is necessary and sufficient that the function*

$$G(z) = \sum_{n=0}^{\infty} a_n^{\rho/\nu} c_n z^n$$

be an entire function of the class $[\rho, \sigma(\sigma\rho/\tau\nu)^{\rho/\nu}]$.

We note that if $\nu = \rho$, then the condition of the theorem is equivalent to the requirement of regularity in the disk $|z| < (\tau/\sigma)^{1/\rho}$ of the function

$$\varphi(z) = \sum_{n=0}^{\infty} c_n z^n,$$

which is called the characteristic function for the operator $M(F)$.

In the case when $\nu < 0$, the series $\sum_{n=0}^{\infty} c_n z^n$ may diverge everywhere except at the point $z = 0$. Consequently, in this case the characteristic function may fail to exist altogether.

The following two theorems indicate the growth of the operator $M(F)$ under the assumption that $F(z)$ is an entire function of finite order and normal type.

Theorem 2. Let $F(z)$ be an entire function of finite order ν, of normal type τ, and let the numbers $c_0, c_1, ..., c_n, ...$ be such that the function

$$G(z) = \sum_{n=0}^{\infty} a_n^{\rho/\nu} c_n z^n$$

is an entire function of class $\left[\rho,\ \sigma(\sigma\rho/\tau\nu)^{\rho/\nu}\right]$.

Then the following assertions hold:

1. $h(z) = M[F(z)]$ is an entire function.
2. If $0 < \nu \leqslant \rho$, then $h(z) \in [\nu, \tau]$.
3. If $\nu > \rho$ and $G(z) \in [(\rho, 0)]$, then $h(z) \in [(\nu, \tau)]$.
4. If $\nu > \rho$ and $G(z)$ is an entire function of order ρ and type $\lambda < \sigma(\sigma\rho/\tau\nu)^{\rho/\nu}$, then $h(z) \in [\nu, d]$, where

$$d = \tau \left\{ 1 - \frac{(\rho\lambda)^{1/\rho}(\tau\nu)^{1/\nu}}{(\rho\sigma)^{1/\rho}(\rho\sigma)^{1/\nu}} \right\}^{\rho\nu/(\nu-\rho)} {}_{(\rho-\nu)/\rho}.$$

In the case $0 < \nu < \rho$, regarding the growth of the operator one can prove a more precise assertion.

Theorem 3. Let $F(z)$ be an entire function of normal type τ, of finite order ν, where $0 < \nu < \rho$. Let, moreover, the numbers $c_0, c_1, \dots, \dots, c_n, \dots$ not all be zero and be such that the function

$$G(z) = \sum_{n=0}^{\infty} a_n^{\rho/\nu} c_n z^n$$

is entire, belonging to the class $\left[\rho,\ \sigma(\sigma\rho/\tau\nu)^{\rho/\nu}\right]$.

Then the function $h(z) = M(F)$ is entire, of order ν and type τ.

Consider the equation

$$M(F) = h(z), \tag{3}$$

where $h(z)$ is an entire function of order less than ρ. For equation (3) the following holds.

Theorem 4. Let $h(z)$ be an entire function of normal type τ, of order ν, where $0 < \nu < \rho$. Let, further, the operator $M(F)$ be such that $c_0 \neq 0$ and the function

$$G(z) = \sum_{n=0}^{\infty} a_n^{\rho/\nu} c_n z^n$$

is entire, belonging to the class $\left[\rho,\ \sigma(\sigma\rho/\tau\nu)^{\rho/\nu}\right]$.

Then equation (3) has one and only one solution in the class $[\nu, \tau]$, and the order and type of the solution are exactly equal, respectively, to ν and τ.

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CITED LITERATURE

[1] R. C. Sikkema, *Differential Operators and Differential Equations of Infinite Order with Constant Coefficients*, Groningen, 1953. [2] A. O. Gelfond, A. F. Leont' ev, *Matem. sborn.*, **29** (71), 3 (1951).

Note: Figure translations are in progress. See original paper for figures.