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Abstract

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CONFORMAL MAPPINGS OF A MULTIPLY CONNECTED DOMAIN ONTO MULTI-SHEETED SURFACES WITH RECTILINEAR SLITS

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Here the extremal properties of certain mappings of a finitely connected domain onto multi-sheeted Riemann surfaces with rectilinear slits are established.

Let G be a bounded n-connected ($n \geq 2$) domain of the z-plane with boundary consisting of simple closed analytic curves C_μ, $\mu = 1, \dots, n$; let ζ_j, $j = 1, \dots, s$, be arbitrary distinct points of the domain G; let α_j, $\alpha_{0,j}$, $j = 1, \dots, s$, be arbitrary coefficients with $\sum_{j=1}^{s} \alpha_{0,j} = 0$, but not all equal to zero;

$$S(z; \zeta, \alpha) = \sum_{j=1}^{s} \left[\frac{\alpha_j}{z - \zeta_j} + \alpha_{0,j} \log(z - \zeta_j) \right]$$

is the singularity function; $\theta \in (-\pi/2,\ \pi/2]$; n_1, n_2 are arbitrary nonnegative integers with $n_1 + n_2 \leq n - 1$.

The existence is proved of functions $\Phi_{k,\theta}(z) = S(z; \zeta, \alpha) + F_{k,\theta}(z)$, $k = 1, 2$, with functions $F_{k,\theta}(z)$ regular in the closed domain G, possessing the following properties: on each boundary component C_μ of the domain G, each branch of the function $e^{-i\theta}\Phi_{k,\theta}(z)$, $k = 1, 2$, for $\mu = 1, \dots, n_1$, has constant imaginary part; for $\mu = n_1 + 1, \dots, n_1 + n_2$, constant real part; for $\mu = n_1 + n_2 + 1, \dots, n$, on each component C_μ each branch of the function $e^{-i\theta}\Phi_{1,\theta}(z)$ has constant imaginary part, while each branch of the function $e^{-i\theta}\Phi_{2,\theta}(z)$ has constant real part. By these properties each of the functions $\Phi_{k,\theta}(z)$, $k = 1, 2$, is determined uniquely up to an additive constant.

It is also proved that the sequence of functions $\Phi_{k,\theta}^{[\nu]}(z)$, $\nu = 1, 2, \dots$, analogously defined by domains $G^{(\nu)}$, as $\nu \to \infty$ approximating the domain G from within, converges uniformly inside this domain to the function $\Phi_{k,\theta}(z)$, $k = 1, 2$.

Consider the class $\mathcal{L}_\theta(G; S; n_1, n_2)$ of all functions $f(z)$ with the difference $f(z) - S(z; \zeta, \alpha)$ regular in the domain G and on its boundary components C_μ, $\mu = 1, \dots, n_1 + n_2$, assigning to these boundary components, for $\mu = 1, \dots, n_1$, segments of inclination θ to the real axis, and for $\mu = n_1 + 1, \dots, n_1 + n_2$, segments perpendicular to them (the precise meaning of this property was indicated above as applied to the functions $\Phi_{k,\theta}(z)$, $k = 1, 2$). For $n_1 = n_2 = 0$ this class does not depend on θ and is the class of all functions $f(z)$ with the difference $f(z) - S(z;\zeta,\alpha)$ regular in the domain G.

Let θ and the singularity function $S(z;\zeta,\alpha)$ be given. Then on the class of all functions of the form

$$f(z) = \sum_{j=1}^{s} \left[\frac{\gamma_j}{z-\zeta_j} + \gamma_{0,j} \log(z-\zeta_j) \right] + F(z),$$

where $\{\gamma\}$ are arbitrary coefficients and $F(z)$ is an arbitrary function regular in the domain G, the functional

$$I_\theta(f) = \operatorname{Re}\left\{ e^{-2i\theta} \sum_{j=1}^{s} \left[\alpha'_{jF}(\zeta_j) - \alpha_{0,j} F(\zeta_j) \right] \right\}$$

is defined.

For a function $f(z) \in \mathcal{L}_\theta(G;S;n_1,n_2)$, denote by $\bar{A}(f)$ the exterior area of the function $f(z)$ in the domain G (see ([1,2])).

Theorem 1. *For any given θ and any prescribed nonnegative β and γ with $\beta+\gamma=1$, in the class $\mathcal{L}_\theta(G;S;n_1,n_2)$ we have the sharp estimate:*

$$\bar{A}(f) + 2\pi(\beta-\gamma)I_\theta(f) \le 2\pi \left[\beta^2 I_\theta(\Phi_{1,\theta}) - \gamma^2 I_\theta(\Phi_{2,\theta}) \right],$$

where the equality sign is attained for the function $f = \beta\Phi_{1,\theta} + \gamma\Phi_{2,\theta}$, up to an additive constant, and only for it.

For $\theta = 0$ and $S(z;\zeta,\alpha) = 1/(z-\zeta)$ we obtain Jenkins' theorem [3], proved by him under the additional assumption of regularity of the function $f(z)$ on the entire boundary of the domain G.

Put

$$Q_\theta(z) = \frac{1}{2}[\Phi_{1,\theta}(z) + \Phi_{2,\theta}(z)].$$

Corollary. Among all functions of the class $\mathcal{L}_\theta(G;S;n_1,n_2)$, the greatest exterior area in the domain G is given by the function $Q_\theta(z)$, and, up to an additive constant, only by it.

For $n_1 = n_2 = 0$ the function $Q_\theta(z)$ does not depend on θ, and in this case we obtain the already known result [1,2].

Put

$$P_\theta(z) = \frac{1}{2}[\Phi_{1,\theta}(z) - \Phi_{2,\theta}(z)]$$

and, for any function $f(z)$ regular in the domain G, denote by $A(f)$ the area of the image of the domain G under the mapping $w = f(z)$. We have

$$A(P_\theta) = A(Q_\theta) = \pi I_\theta(P_\theta).$$

For any prescribed θ and any prescribed function $S(z;\zeta,\alpha)$, consider the class $R_\theta(G;S;n_1,n_2)$ of all functions $f(z)$ regular in the domain G and on its boundary components C_μ, $\mu = 1,\dots,n_1+n_2$, which assign to the boundary components C_μ of this domain, for $\mu = 1,\dots,n_1$, segments of inclination θ to the real axis, and for $\mu = n_1+1,\dots,n_1+n_2$, segments perpendicular to them, and which are normalized by the condition $I_\theta(f) = 1$. $R_\theta(G;S;0,0)$ is the class of all functions regular in the domain G and normalized in the indicated manner.

Theorem 2. Among all functions of the class $R_\theta(G;S;n_1,n_2)$, the least area of the image of the domain G is given by the function $P_\theta(z)/I_\theta(P_\theta)$, and, up to an additive constant, only by it.

Putting $\theta = 0$, $S(z;\zeta,\alpha) = 1/(z-\zeta)$, we obtain the solution of the problem of the least area of the image of the domain G in the class of all functions $f(z)$ regular in this domain, assigning to its boundary components C_μ horizontal (for $\mu = 1,\dots,n_1$) and vertical (for $\mu = n_1+1,\dots,n_1+n_2$) segments and normalized by the condition $\operatorname{Re} f'(\zeta) = 1$, $\zeta \in G$. Putting $\theta = 0$, $S(z;\zeta,\alpha) = \log(z-\zeta_1)/(z-\zeta_2)$, $\zeta_1,\zeta_2 \in G$, we find the solution of this problem in the same class of regular functions, but normalized by the conditions: $f(\zeta_1) = 0$, $\operatorname{Re} f(\zeta_2) = 1$.

Let us also note the following consequence of Theorems 1 and 2.

Corollary. The product of the least area in the minimal problem considered for the class $R_\theta(G;S;n_1,n_2)$ and the greatest exterior area in the maximal problem considered for the class $\mathcal{L}_\theta(G;S;n_1,n_2)$ is equal to π^2.

This generalizes, to the indicated classes of functions, the well-known [4] relation between the least area of the image of the domain G in the class of all functions regular in it with $f'(\zeta) = 1$, $\zeta \in G$, and the greatest area of the complement to the image of this domain in the class of all univalent functions $f(z)$ in it with regular difference $f(z) - [1/(z-\zeta)]$.

For a doubly connected domain G, all results generalize to the case of the singularity function

$$S(z;\zeta,\alpha) = \sum_{j=1}^{s} \left[\sum_{k=1}^{p_j} \frac{\alpha_{k,j}}{(z-\zeta_j)^k} + \alpha_{0,j} \log(z-\zeta_j) \right]$$

and of the functional

$$I_\theta(f) = \operatorname{Re} \left\{ e^{-2i\theta} \sum_{j=1}^{s} \left[\sum_{k=1}^{p_j} \frac{\alpha_{k,j}}{(k-1)!} F^{(k)}(\zeta_j) - \alpha_{0,j} F(\zeta_j) \right] \right\}.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.