MATHEMATICS

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1965

SovietRxiv

Abstract

Full Text

MATHEMATICS

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ASYMPTOTIC BEHAVIOR AS $t \to \infty$ OF THE SOLUTION OF THE NONSTATIONARY SCHRÖDINGER EQUATION WITH A NON-SELF-ADJOINT HAMILTONIAN

(Presented by Academician I. N. Vekua, 22 IX 1964)

1. In a Hilbert space $\mathscr{H}$, consider the nonstationary Schrödinger equation

$$i\frac{\partial \psi(t)}{\partial t} = H\psi(t) \qquad (-\infty < t < \infty) \tag{1}$$

with the initial condition

$$\psi(0) = f,$$

where f belongs to the domain of definition $D(H)$ of the closed operator H.

For physical applications, it is of interest to study the behavior of the solution of equation (1) for large values of the parameter t. In particular, if the operator H is self-adjoint and can be regarded as the result of a perturbation of a self-adjoint operator H_0, $H = H_0 + V$, and if the initial condition belongs to the absolutely continuous subspace with respect to the operator H, then, for a broad class of operators, it has been shown that for large values of t ($t = \pm\infty$) the solution of the Schrödinger equation with Hamiltonian H behaves, in the sense of the metric of the space $\mathscr{H}$, like the solution of the Schrödinger equation with the unperturbed Hamiltonian H_0 under the initial condition g, obtained from f by means of a linear operator $U_\pm$, $g = U_\pm f$. The latter are called wave operators. With their aid the scattering operator S, $S = U_+ U_-^*$, is constructed, which plays an important role in physics. Moreover, the wave operators satisfy the important relation

$$UHf = H_0 Uf$$

for every f in $D(H)$.

In connection with what has been said, we note the works ($^{1-4}$), which are devoted to these questions*.

In the present article the asymptotic behavior of the solution of the nonstationary Schrödinger equation is studied for the case of certain non-self-adjoint Hamiltonians H.

2. **Definition 1.** We shall call a closed operator H **linearly similar** to an operator H_0 if there exists a bounded operator U such that, for every f in $D(H_0)$,

$$HUf = UH_0f.$$

First of all, we shall indicate two sufficient conditions under which a perturbed operator turns out to be linearly similar to an unperturbed operator.

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* These works also contain an extensive bibliography.

Definition 2. We shall call a closed operator H **dissipative** if:

1) $\operatorname{Im}(Hf, f) \geqslant 0$ $(f \in D(H))$;
2) for every $\lambda > 0$ there exists a bounded operator $(iH + \lambda)^{-1}$, and moreover

$$\left\|(iH + \lambda)^{-1}\right\| \leqslant 1/\lambda.$$

We indicate a condition under which the perturbed operator turns out to be dissipative.

Lemma 1. *Let H_0 be self-adjoint and let V be such a closed operator that:*

1) $D(V) \supset D(H_0)$;
2) $\operatorname{Im}(Vf, f) \geqslant 0$ for every $f \in D(H_0)$;
3) *there exist constants $0 \leqslant a < 1$, $b > 0$ such that, for every f from $D(H_0)$,*

$$\|Vf\| \leqslant a\|H_0f\| + b\|f\|.$$

Then the operator $H = H_0 + V$, defined on $D(H_0)$, is dissipative.

From Definition 2, by virtue of theorems from the theory of semigroups of linear operators, it follows that the operator iH, satisfying Lemma 1, is the infinitesimal operator of a strongly continuous semigroup $T(H, t)$, $t \geqslant 0$, and in addition $\|T(H, t)\| \leqslant 1$.

Denote by $\mathcal{P}_0$ the projector onto the absolutely continuous subspace of the operator H_0. Then the following holds.

Theorem 1. *Let the operators H_0 and V satisfy the conditions of Lemma 1 and, moreover:*

1) *there exists a set M with linear span dense in $\mathcal{P}_0\mathcal{H}$, for each element f of which*
$$I(f) = \int_0^\infty \left\| V e^{-itH_0} f \right\| dt < \infty; \tag{2}$$

2) *there exists at least one element g from $D(H_0)$ such that*
$$I(g) < \tfrac{1}{2}\|g\|.$$

Then the operator H is linearly similar to the operator H_0.

We note that from condition 1 there follows the existence of an operator U such that
$$U = \lim_{t\to\infty} T(H,t) e^{-itH_0} \mathcal{P}_0$$
in the sense of strong convergence. In addition, it turns out that $HUf = UH_0 f$ for every f from $D(H_0)$. From condition 2 of the theorem it follows that the operator U obtained in this way is not identically zero: in any case $Ug \neq 0$.

This theorem generalizes the result obtained by Jauch and Zinnes (^{8}) in the case of self-adjoint operators H_0 and H.

Example 1. Let $\mathcal{H} = L_2(E_k)$ and
$$H_0 f = -\sum_{j=1}^{k} \frac{\partial^2}{\partial x_j^2} f$$
with domain of definition
$$D(H_0) = \left\{ f,\ f = \int_{E_k} \frac{\varphi(p) e^{i(p,x)}}{p^2+1}\, d^{kp}, \quad \varphi(p) \in L_2(E_k) \right\}.$$

Let the potential $V(x)$ satisfy the following conditions
$$\operatorname{Im} V(x) \geqslant 0; \tag{3}$$
$$|V(x)| < c < \infty; \tag{4}$$
$$V(x)(1+|x|)^{-(k/2-1)-\varepsilon} \in L_2(E_k) \qquad (1/4 > \varepsilon > 0). \tag{5}$$

In this case the conditions of Theorem 1 for the operators H_0 and V, $Vf = V(x)f(x)$, are satisfied.

Let us now consider the space $\mathcal{H} = L_2(-\infty, \infty, \mathfrak{B})$ of abstract functions of the real variable x, $-\infty < x < \infty$, with values in the Hilbert space $\mathfrak{B}$.

Define the operator
$$H = H_0 + \varepsilon P,$$

where H_0 is the self-adjoint operator of multiplication by x in the space $\mathcal{H}$ with domain of definition $D(H_0) = \{f,\ f \in \mathcal{H},\ xf \in \mathcal{H}\}$; ε is a real number; P is a bounded integral operator with kernel $p(x, y)$, which satisfies the following conditions:

$$\begin{gathered}
\sup_{x,y}(1+|x|^{\gamma})|p(x,y)|(1+|y|^{\gamma}) < \infty, \\
\sup_{x,y,h>0}(1+|y|^{\gamma})|p(x+h,y)-p(x-h,y)|h^{-\beta} < \infty, \\
\sup_{x,y,h'>0}(1+|x|^{\gamma})|p(x,y+h')-p(x,y-h')|h'^{-\beta} < \infty, \qquad (6) \\
\sup_{x,y,h>0,\ h'>0}|p(x+h,y+h')-p(x+h,y-h')+p(x-h,y+h') \\
-p(x-h,y-h')|\,h^{-\beta}h'^{-\beta} < \infty,
\end{gathered}$$

where $\gamma > 1/2$ and $0 < \beta < 1$.* The operator H under these assumptions was considered in works (^{5}–^{7}).

O. A. Ladyzhenskaya and L. D. Faddeev in (^{7}) showed that if it is assumed that the operator P is self-adjoint, then the operator H is linearly similar to the operator H_0. The case of a non-self-adjoint operator P was considered by Friedrichs (^{5},^{6}), who proved an analogous assertion in the case where ε is sufficiently small in absolute value. We generalize this result to the case of arbitrary ε.

Theorem 2. *Let P satisfy conditions (6); then the operator H is linearly similar to the operator H_0.*

This theorem is applicable to the Schrödinger operator in the space $L_2(E_k)$, $k \geqslant 3$, with complex potential $V(x)$ satisfying the condition: all derivatives up to order $(n-1)$ exist and are such that

$$\int_{E_k}(1+|x|^2)\left|\frac{\partial^j V(x)}{\partial x_1^{j_1}\dots\partial x_k^{j_k}}\right| d^k x < \infty \qquad (0 \le j \le n-1). \qquad (7)$$

3. Let H_0 be a self-adjoint operator. Denote by $W(H_0)$ the set of closed operators V such that:

1) $D(V) \supset D(H_0)$; $D(V^*) \supset D(H_0)$;
2) $H = H_0 + V$ and $H^* = H_0 + V^*$ are closed on $D(H_0)$;
3) there exists a bounded operator U such that, for f in $D(H_0)$,

$$HUf = UH_0 f.$$

It follows from the preceding that there exist operators H_0 for which this set is nonempty, and $W(H_0)$ contains non-self-adjoint operators.

Let $V \in W(H_0)$. Consider problem (1) with the initial condition $f = Ug$, where $g \in D(H_0)$. Then there exists a solution of this problem which can be represented in the form

$$\psi(t) = Ue^{itH_0} g.$$

The asymptotics of the solution $\psi(t)$ as $t \to \infty$ are given by the following

Theorem 3. *Let H_0 be a self-adjoint operator such that $\mathfrak{P}_0 \mathcal{H} = \mathcal{H}$, and, moreover, let V be a closed operator such that*

1) $V \in W(H_0)$;

2) *there exist an integer n and a point z, $z = \sigma + i\tau$, such that:*

a) $\|(H_0^n - z)(H^n - z)^{-1}\| < \infty$;

b) *the operators $|V|^{1/2}(H_0^n - z)^{-1}$ and $|V^*|^{1/2}(H_{0n} - z)^{-1}$ are operators of Hilbert-Schmidt type.*

Then there exists a bounded operator $S_\pm$ such that, for every f from $D(H_0)$,

$$\lim_{t \to \pm\infty} \|\psi(t) - e^{itH_0} S_\pm f\| = 0;$$

moreover,

$$H_0 S_\pm f = S_\pm H_0 f.$$

Example 2. Let H_0 and V be the same as in Example 1, and, in addition,

$$V(x) \in L_1(E_k);$$

then the conditions of Theorem 3 are satisfied.

Theorem 3 is also applicable to the Schrödinger operator with a potential satisfying condition (7).

Finally, let us formulate condition 2) of Theorem 3 in terms of the operators H_0 and V.

Lemma 2. *Let H_0 be self-adjoint and V a closed operator such that:*

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* For fixed x and y, $p(x, y)$ is an operator in the space $\mathfrak{B}$. Its norm in the space $\mathfrak{B}$ is denoted by $|p(x, y)|$.

1) *$D(V) \supset D(H_0)$;*
2) *there exists an integer n such that, for every integer l, $0 \le l \le n-1$, and every f from $D(H_0^{l+1})$:*

a) *$Vf \in D(H_0^l)$;*

b)

$$\|H_0^{lVf}\| \le a\|H_0^{l+1}f\| + b\sum_{j=0}^{l}\|H_0^{jf}\|; \qquad 0 \le a < (1+2^n)^{-1}; \quad b > 0.$$

Then:

1) *the operator $H = H_0 + V$ on $D(H_0)$ is closed;*
2) *there exists at least one point $z = \sigma + i\tau$ at which the operator $(H_0^n - z)(H^n - z)^{-1}$ is bounded.*

In conclusion, the author expresses his sincere gratitude to F. A. Berezin for posing the problem and for constant attention to this work, and also to M. A. Naimark and A. G. Kostyuchenko for discussion of the results obtained.

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Received
16 IX 1964

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Note: Figure translations are in progress. See original paper for figures.

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