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Abstract

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MATHEMATICS

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POINCARÉ POLYNOMIALS OF SOME COMPACT HOMOGENEOUS SPACES

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1°. Let $\mathcal{G}$ be a connected compact Lie group; $\mathcal{T}$ its maximal torus; G, T the corresponding Lie algebras; $I_{S(G)}$ the algebra of all polynomials on T invariant with respect to the Weyl group. Then $I_{S(G)}$ is a free algebra with r (r is the rank of $\mathcal{G}$) algebraically independent homogeneous generators $P_{k_1}, P_{k_2}, \dots, P_{k_r}$ (k_i is the degree of P_{k_i}), and the Poincaré polynomial of the group $\mathcal{G}$ has the form

$$P(\mathcal{G}, t) = (1 + t^{2k_1-1})(1 + t^{2k_2-1}) \cdots (1 + t^{2k_r-1}).$$

Theorem. *Let G be a compact exceptional simple Lie algebra; let $p_0(G)$ be its linear representation of least dimension; denote the weights of the representation $p_0(G)$ by $\Lambda_1 \Lambda_2, \dots, \Lambda_n$ (n is the dimension of p_0).*

Then

$$P_{k_i} = \sum_{\alpha=1}^{n} \Lambda_\alpha^{k_i}$$

for the following k_i form generating polynomials of the algebra $I_{S(G)}$ (in parentheses are the values of k_i):

$$G_2(2, 6); \quad F_4(2, 6, 8, 12); \quad E_6(2, 5, 6, 8, 9, 12);$$

$$E_7(2, 6, 8, 10, 12, 14, 18); \quad E_8(2, 8, 12, 14, 18, 20, 24, 30).$$

(These P_{k_i} coincide with the polynomials considered by Coxeter ([1]) for E_6 and by Takeuchi ([2]) for F_4.)

We note that an analogue of this theorem, as is known, is also valid for A_r, B_r, C_r. For D_r, besides

$$P_{k_i} = \sum_{\alpha=1}^{2r} \Lambda_\alpha^{k_i},$$

one must also take $P'_r = \Lambda_1 \Lambda_2 \cdots \Lambda_r$.

2°. Let a homogeneous space $\mathcal{G}/\mathcal{U}$ be given, where $\mathcal{G}$ is a connected compact Lie group, $\mathcal{U}$ its connected closed subgroup; $U \subset G$ are the corresponding Lie algebras. Denote the restriction of $I_{S(G)}$ to U by $p : I_{S(G)} \to I_{S(U)}$. If the rank of $\mathcal{G} = R$, the rank of $U = r$, and all $p(P_{k_\alpha})$ $(r+1 \leq \alpha \leq R)$ belong to the ideal in $I_{S(U)}$ generated by $p(P_{k_i})$ $(1 \leq i \leq r)$ (where $P_{k_1}, P_{k_2}, \dots, P_{k_R}$ are generating polynomials of the algebra $I_{S(G)}$), then we shall say that $\mathcal{U}$ is in normal position in $\mathcal{G}$. In this case the Poincaré polynomial of the space $\mathcal{G}/\mathcal{U}$ has the form

$$P(\mathcal{G}/\mathcal{U}, t) = \frac{(1-t^{2k_1})(1-t^{2k_2})\cdots(1-t^{2k_r})}{(1-t^{2l_1})(1-t^{2l_2})\cdots(1-t^{2l_r})} \times$$

$$\times (1+t^{2k_{r+1}-1})(1+t^{2k_{r+2}-1})\cdots(1+t^{2k_R-1}),$$

where $l_1, l_2, \dots, l_r$ are the degrees of the generating polynomials of the algebra $I_{S(U)}$.

The case $R = r$ (then we have Hirzebruch' s formula) or the case $p : I_{S(G)} \to I_{S(U)}$ is an epimorphism (then $\mathcal{U}$ is completely nonhomologous to 0 in $\mathcal{G}$ and $P(\mathcal{G}/\mathcal{U}, t) = \dfrac{P(\mathcal{G}, t)}{P(\mathcal{U}, t)}$) is a special case of normal position.

3°. Consider compact homogeneous nonsymmetric spaces $\mathfrak{G}/\mathfrak{U}$ with an irreducible rotation group. All these spaces were found in the papers of O. V. Manturov ([3a,b]).

For these spaces $\mathfrak{U}$ is in normal position in $\mathfrak{G}$, except for the following 4 cases (in the diagrams the embedding $U \to G$ is indicated):

a)

$$\overset{1}{\circ} - \underbrace{\circ - \cdots - \circ}_{n_1} \times \overset{1}{\circ} - \underbrace{\circ - \cdots - \circ}_{n_2} \to SU\big((n_1+1)(n_2+1)\big)$$

or

$$\overset{1}{\circ} - \underbrace{\circ - \cdots - \circ \times \circ}_{n_1} - \underbrace{\circ - \cdots - \overset{1}{\circ}}_{n_2} \to SU\big((n_1+1)(n_2+1)\big)$$

(where $\times$ denotes the Kronecker product);

b)

$$\underbrace{\overset{1}{\circ} - \circ - \cdots - \overset{1}{\circ}}_{n} \to SO(n(n+2));$$

c)

$$\circ - \circ - \overset{1}{\circ} - \circ - \circ \to Sp(20);$$

d)

$$\circ - \circ - \begin{array}{c} \circ \\ | \\ \overset{1}{\circ} \end{array} - \circ - \circ \to SO(78).$$

Below the Poincaré polynomials of all these spaces are obtained, except for cases a), b).

Let us note that for all symmetric homogeneous spaces $\mathfrak{G}/\mathfrak{U}$, $\mathfrak{U}$ is always in normal position in $\mathfrak{G}$ (see ([4])). The Poincaré polynomials of all irreducible symmetric spaces $\mathfrak{G}/\mathfrak{U}$ were obtained by Takeuchi ([2]).

4°. Consider n variables $x_1, x_2, \dots, x_r$ and the polynomials

$$p_\alpha = \sum_{i=1}^{n} x_i^\alpha \qquad (\alpha \text{ an integer } > 0);$$

then

$$p_\alpha = Q_\alpha(p_1, p_2, \dots, p_n).$$

If in Q_{ms} (m an integer > 0; $0 < s$ an integer $< n$) all arguments are set equal to 0 except for $p_s, p_{2s}, \dots, p_{i_0 s}$ ($i_0 = [n/s]$), then the expression obtained will be denoted by

$$Q^s_{ms}(p_s, p_{2s}, \dots, p_{i_0 s}).$$

Then the Poincaré polynomials of the preceding spaces have the form:

A. The case $\mathfrak{U}$ is not in normal position in $\mathfrak{G}$ (see 3°) (for cases a), b) the Poincaré polynomial for large n_1, n_2, n has not yet been obtained):

c)

$$P(\mathfrak{G}/\mathfrak{U}, t) = (1 + t^6 + t^{10} + t^{12} + t^{18} + t^{25} + t^{31} + t^{33} + t^{37} + t^{43})$$
$$\times (1 + t^{27})(1 + t^{31})(1 + t^{35})(1 + t^{39});$$

d)

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 + t^9)(1 + t^{17})}{(1 + t^{19})(1 + t^{27})(1 + t^{35})} (1 + t^{10} + t^{18} + t^{37} + t^{45} + t^{55}).$$

B. $\mathfrak{U}$ is in normal position in $\mathfrak{G}$ (but is not wholly homologous to 0 in $\mathfrak{G}$):

a)

$$\underbrace{\circ - \circ - \cdots - \circ \to SU\left(\frac{n(n-1)}{2}\right)}_{n-1}, \qquad n = 2^{s-1};$$

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1-t^{2ps})}{(1-t^{2s})} \frac{(1+t^{2s-1})}{(1+t^{2ps-1})},$$

where p is determined as follows: put

$$A_1 = 1, \qquad A_i = -\frac{1}{2n-2^{is}} \sum_{\alpha=1}^{i-1} C_{is}^{\alpha s} A_\alpha A_{i-\alpha} \qquad \left(i = 1, 2, \dots, i_0 = \left[\frac{n}{s}\right]\right);$$

$$A_m = Q_{ms}^s(A_1, A_2, \dots, A_{i_0}).$$

Then p is the smallest integer $> i_0$ such that

$$(2n - 2^{ps})A_p + \sum_{\alpha=1}^{p-1} C_{ps}^{\alpha s} A_\alpha A_{p-\alpha} \neq 0.$$

b)

$$\underbrace{\bullet - \overset{1}{\bullet} - \cdots - \bullet = \circ \to SO((n-1)(2n+1))}_{n}, \qquad n = 2^{2s-2};$$

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1-t^{4ps})}{(1-t^{4s})} \frac{(1+t^{4s-1})}{(1+t^{4ps-1})},$$

where p is defined as follows: put

$$A_1 = 1, \qquad A_i = \frac{-1}{2n-2^{2is-1}} \sum_{\alpha=1}^{i-1} C_{2is}^{2\alpha s} A_\alpha A_{i-\alpha} \qquad \left(i = 1, 2, \dots, i_0 = \left[\frac{n}{s}\right]\right);$$

$$A_m = Q_{ms}^s(A_1, A_2, \dots, A_{i_0}).$$

Then p is the least integer $> i_0$ such that

$$(2n - 2^{2ps-1})A_p + \sum_{\alpha=1}^{p-1} C_{2ps}^{2\alpha s} A_\alpha A_{p-\alpha} \neq 0.$$

c) $\circ - \overset{1}{\circ} - \circ - \cdots - \circ\!\begin{smallmatrix} \diagup \circ \\ \diagdown \circ \end{smallmatrix} \to SO(n(2n-1)):$

$$\underbrace{\qquad\qquad\qquad\qquad}_{n}$$

1) $n \neq 2^{2s-2}$,

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 + t^{2n-1})}{(1 + t^{4n-1})} (1 + t^{2n});$$

2) $n = 2^{2s-2}$,

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 + t^{2n-1})}{(1 + t^{4n-1})} \frac{(1 + t^{4s-1})}{(1 + t^{4ps-1})} (1 + t^{2n}) \frac{1 + t^{4ps}}{1 - t^{4s}},$$

where p is defined as in case b).

d) $\overset{1}{\circ} \times \overset{1}{\circ} - \circ - \cdots - \circ = \bullet \to Sp(4n+2)$;

$$\underbrace{\qquad\qquad\qquad\qquad}_{n}$$

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 - t^{4p})}{(1 - t^{4})} \frac{(1 + t^{3})}{(1 + t^{4p-1})},$$

where p is defined as follows: put

$$A_0 = n, \qquad A_i = -1 - \sum_{\alpha=0}^{i-1} C_{2i}^{2\alpha} A_\alpha \quad (i = 1, 2, \dots, n);$$

$$A_m = Q_m(A_1, A_2, \dots, A_n).$$

Then p is the least integer $> n$ such that

$$A_p + \sum_{\alpha=0}^{n-1} C_{2p}^{2\alpha} A_\alpha + 1 \neq 0.$$

e) $\overset{1}{\circ} \times \overset{1}{\circ} - \cdots - \circ\!\!\begin{array}{l}\diagup\,\circ \\ \diagdown\,\circ\end{array} \to Sp(4n)$;

$$\underbrace{\qquad\qquad\qquad\qquad}_{n}$$

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)(1 + t^{2n-1})}{P(\mathfrak{U}, t)(1 + t^{4n-1})} \frac{(1 + t^{3})}{(1 + t^{4p-1})} (1 + t^{2n}) \frac{(1 - t^{4p})}{(1 - t^{4})},$$

where p is defined as follows: put

$$A_0 = n, \qquad A_i = -\sum_{\alpha=0}^{i-1} C_{2i}^{2\alpha} A_\alpha \quad (i = 1, 2, \dots, n);$$

$$A_m = Q_m(A_1, A_2, \dots, A_n).$$

Then p is the least integer $> n$ such that

$$A_p + \sum_{\alpha=0}^{p-1} C_{2p}^{2\alpha} A_\alpha \neq 0.$$

f) $\overset{1}{\circ} \times \underbrace{\overset{1}{\bullet} - \bullet - \cdots - \bullet}_{n} = \circ \to SO(4n),$

Set

$$A_0 = n, \quad A_i = -\sum_{\alpha=0}^{i-1} C_{2i}^{2\alpha} A_\alpha, \quad B_i = \sum_{\alpha=0}^{i} (-1)^\alpha C_i^\alpha C_{2i}^\alpha A_\alpha \quad (i = 1, \dots, n).$$

Let the function $R(p_1, p_2, \dots, p_n)$ express $x_1 x_2 \cdots x_n$ in terms of $p_1, p_2, \dots, p_n$, $(p_\alpha = \sum_{i=1}^n x_i^\alpha)$.

1) If $R(B_1, B_2, \dots, B_n) \neq 0$, then

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 - t^{4n})}{(1 - t^4)} \frac{(1 + t^3)}{(1 + t^{4n-1})}.$$

2) If $R(B_1, B_2, \dots, B_n) = 0$, then

$$P(\mathfrak{G}/\mathfrak{U}, t) = \frac{P(\mathfrak{G}, t)}{P(\mathfrak{U}, t)} \frac{(1 + t^3)}{(1 + t^{4p-1})} \frac{(1 - t^{4p})}{(1 - t^4)},$$

where p is determined as follows: set $A_m = Q_m(A_1, A_2, \dots, A_n)$; then p is the least integer $> n$ such that

$$A_p + \sum_{\alpha=0}^{p-1} C_{2p}^{2\alpha} A_\alpha \neq 0.$$

) $\overset{2}{\circ} - \circ - \cdots - \circ\!\!\begin{array}{c}\circ\\ |\\ \circ\end{array} \to SO((n+1)(2n-1));$

$$P(\mathfrak{G}/\mathfrak{U},t) = \frac{P(\mathfrak{G},t)}{P(\mathfrak{U},t)}\frac{(1+t^{2n-1})}{(1+t^{4n-1})}(1+t^{2n}).$$

) $\circ - \overset{2}{\circ} - \circ \to Sp(20):$

$$P(\mathfrak{G}/\mathfrak{U},t) = \frac{P(\mathfrak{G},t)}{P(\mathfrak{U},t)}\frac{(1+t^{5})}{(1+t^{11})}(1+t^{6}).$$

$\overset{1}{\circ} \times \overset{2}{\circ} \to Sp(6):$

$$P(\mathfrak{G}/\mathfrak{U},t) = (1+t^4)(1+t^{11}).$$

$\overset{1}{\circ} \times \overset{1}{\circ} \times \overset{1}{\circ} \to Sp(8):$

$$P(\mathfrak{G}/\mathfrak{U},t) = (1+t^4)(1+t^4+t^8)(1+t^{15}).$$

$\overset{1}{\circ} \times \circ - \overset{1}{\circ} - \circ \to Sp(12):$

$$P(\mathfrak{G}/\mathfrak{U},t) = (1+t^4+t^8+t^{12})(1+t^6)(1+t^{19})(1+t^{23}).$$

$\overset{2}{\circ} - \overset{2}{\circ} \to \overset{1}{\circ} - \circ - \circ - \circ - \circ - \circ :$

$$P(\mathfrak{G}/\mathfrak{U},t) = \frac{P(\mathfrak{G},t)}{P(\mathfrak{U},t)}\frac{1+t^{5}}{1+t^{11}}(1+t^{6}).$$

$\overset{6}{\circ} - \circ + \circ - \overset{6}{\circ} \to \circ - \circ - \circ - \circ - \circ - \overset{1}{\circ} :$

$$P(\mathfrak{G}/\mathfrak{U},t) = \frac{P(\mathfrak{G},t)}{P(\mathfrak{U},t)}\frac{1+t^{5}}{1+t^{11}}(1+t^{6}).$$

. In the remaining cases $\mathfrak{U}$ is completely nonhomologous to 0 in $\mathfrak{G}$.

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Note: Figure translations are in progress. See original paper for figures.