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Abstract

Full Text

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A Representation Theorem for a Class of Functions

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In the present paper a class of differentiable functions is introduced, defined on the n-dimensional space, whose generalized derivatives of a certain order have a finite mixed norm, while the higher derivatives satisfy a multiple Hölder condition in the same norm. For this class a representation theorem is proved in the form of a series in entire functions satisfying certain conditions.

Let R_n be the n-dimensional space of points $x = (x_1, \dots, x_n)$, $-\infty < x_i < \infty$ $(i = 1, \dots, n)$; $e_n = \{1, \dots, n\}$. By e we shall denote any subset of the set e_n, including the empty set; $r = (r_1, \dots, r_n)$, where $r_j \geq 0$ $(j = 1, \dots, n)$; $r^e = (r_1^e, \dots, r_n^e)$, where $r_j^e = r_j$ if $j \in e$, and $r_j^e = 0$ if $j \in e_n - e$; if $\alpha = (\alpha_1, \dots, \alpha_n)$, then $r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n}$. For an integral vector $k = (k_1, \dots, k_n)$ put $D^k = \partial^{|k|} / \partial x_1^{k_1} \cdots \partial x_n^{k_n}$, where $k_i \geq 0$ are integers, $|k| = \sum_{i=1}^n k_i$;

$$\Delta_{h_i}^2 f(x) = f(x_1, \dots, x_i + 2h_i, \dots, x_n) - 2f(x_1, \dots, x_i + h_i, \dots, x_n) + f(x_1, \dots, x_i, \dots, x_n),$$

$\Delta_h^{2\omega^e} f(x)$, where $h = (h_1, \dots, h_n)$, $h_i > 0$ $(i = 1, \dots, n)$, $\omega = (1, \dots, 1)$, denote the second h-differences applied successively n times to the function $f(x)$ with respect to all variables x_i for which $i \in e$; $(k, r) = \sum_{i=1}^n k_i r_i$.

Let now $p = (p_1, \dots, p_n)$, where $1 \leq p_i \leq \infty$ $(i = 1, \dots, n)$. Denote

$$\|f\|_p = \|f\|_{p_1, \dots, p_n} = \{\| \cdots \|(\|f\|_{p_1})\|_{p_2} \cdots \|_{p_{n-1}}\}\|_{p_n}, \tag{1}$$

where

$$\|f\|_{p_i} = \left(\int_{-\infty}^{\infty} |f(x)|^{p_i} \, dx_i \right)^{1/p_i} \qquad (i = 1, \dots, n).$$

If $\|f\|_p < \infty$, then we shall write $f(x) \in L_p(R_n) \equiv L_{(p_1, \dots, p_n)}(R_n)$. If some $p_i = \infty$, then with respect to the variables x_i, instead of the integral (in the Lebesgue sense), we take the essential maximum.

Definition. Let $r = (r_1, \dots, r_n)$, $r_i = \bar{r}_i + \alpha_i$, where $\bar{r}_i$ are nonnegative integers, $0 < \alpha_i \leq 1$ $(i = 1, \dots, n)$. We shall say that a function $f(x)$ belongs to the class $S_p^{(r)} H(R_n)$, $p = (p_1, \dots, p_n)$, if: 1) the function f and its generalized derivatives (in the sense of Sobolev) $D^k f$, where $0 \le k_i \le \bar{r}_i$ $(i = 1, \dots, n)$, are bounded in the sense of the norm (1); 2) the derivatives $D^{\bar{r}^e} f$, $\bar{r} = (\bar{r}_1, \dots, \bar{r}_n)$, for any $e \subseteq e_n$ satisfy the condition

$$\sup_h \left\|\Delta_h^{2\omega^e} D^{\bar{r}^e} f / h^{\alpha^e}\right\|_p = M_p^{(r^e)}(f) < \infty,$$

where $h = (h_1, \dots, h_n)$, $h_i > 0$ $(i = 1, \dots, n)$; $\alpha = (\alpha_1, \dots, \alpha_n)$.

In the class $S_p'^{(r)} H(R_n)$ the norm is introduced by

$$\|f\|_{S_p^{(r)} H(R_n)} = \sum_{e \subseteq e_n} M_p^{(r^e)}(f),$$

where $M_p^{(r^0)}(f) = \|f\|_p$.

We note that this class of functions was considered by us in paper (1). For $p_1 = \dots = p_n$ the class $S_p^{(r)} H(R_n)$ becomes the class $S_{p_1}^{(r)} H(R_n)$, which was introduced and studied by S. M. Nikol' skii (2).

Theorem. In order that a function $f \in S_p^{(r)} H(R_n)$, where $r = (r_1, \dots, r_n)$, $r_i > 0$ $(i = 1, \dots, n)$; $\mathbf{p} = (p_1, \dots, p_n)$, $1 \le p_i \le \infty$ $(i = 1, \dots, n)$, it is necessary and sufficient that it be representable in the form

$$f(x) = \sum_{e \subseteq e_n} \sum_{k^e > 0} Q_{k^e}(x).$$

Here the outer sum of a finite number of terms (series) extends over all possible subsets $e \subseteq e_n$, including the empty set. The inner sum extends over all possible integer nonnegative vectors $k^e = (k_1^e, \dots, k_n^e)$, $k_i^e \ge 0$. The functions $Q_{k^e}(x)$–integral powers $2^{k_j^e}$ in x_j, $j \in e$ (thus, powers 1 in x_i, $j \in e_n - e$)–satisfy the inequalities

$$\|Q_{k^e}\|_p \le M 2^{-(k, r^e)},$$

where M is a constant.

In proving necessity we obtain $M \le c\|f\|_{S_p^{(r)} H(R^n)}$, and in proving sufficiency

$$\|f\|_{S_p^{(r)} H(R_n)} \le cM.$$

This theorem is a generalization of results of S. M. Nikol' skii (2). On the basis of this theorem A. P. Uninskii proved a number of embedding theorems for the class $S_p^{(r)} H(R_n)$.

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1. Ya. S. Bugrov, Reports of the III Siberian Conference on Mathematics and Mechanics, Tomsk Univ. Press, 1964, p. 54.

2. S. M. Nikol' skii, *Siberian Mathematical Journal*, 4, No. 6, 1342 (1963).

Note: Figure translations are in progress. See original paper for figures.