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Abstract

Full Text

Mathematics

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On the Removability of Level Sets of Solutions of Elliptic Equations

(Presented by Academician M. A. Lavrent'ev on 21 VII 1964)

Let, in a domain D in the plane, there be given a uniformly elliptic operator

$$L\varphi = a_{11}(z)\varphi_{xx} + 2a_{12}(z)\varphi_{xy} + a_{22}(z)\varphi_{yy} + b_1(z)\varphi_x + b_2(z)\varphi_y + c(z)\varphi,$$

$$z = (x, y),$$

with coefficients satisfying the Hölder condition. In this note it is proved that a “weak” solution of the equation $L\psi = 0$ is regular. By a “weak” solution we mean here a function $\psi(z) \in C^1$ in D , which is a regular solution of the equation $L\psi = 0$ in a neighborhood of each point $z \in D$ at which $\psi(z) \neq 0$. The author's interest in this question arose in connection with the problem of the structure of a set γ on which it is possible to glue, together with first derivatives, solutions of two elliptic equations which are equal to zero on γ .

Theorem 1. *Let a function $\psi(z)$ with continuous first derivatives be given in a domain D . Suppose that the function $\psi(z)$ is harmonic in a neighborhood of each point $z \in D$ at which $\psi(z) \neq 0$. Then the function $\psi(z)$ is harmonic everywhere in the domain D .*

Proof.

1°. In any domain D_1 , $\overline{D_1} \subset D$, the function $\psi(z)$ cannot assume a maximum (minimum) value if it is not constant in D_1 . Indeed, suppose that at a point $z_0 \in D_1$ the function $\psi(z)$ assumes a maximum value M and is not constant. Then in D_1 there will be found an open disk in which $\psi(z) \neq 0$ and on whose boundary $\psi(z)$ assumes the value M . From Zaremba's lemma it now follows that the function $\psi(z)$ assumes in D_1 values greater than M .

To prove the theorem it would suffice to show that the maximum principle holds for the function $\Phi(z) = \psi(z) + \varphi(z)$, where $\varphi(z)$ is arbitrary harmonic. An attempt to prove the maximum principle in this more general form leads to the following assertion.

Let, in a domain D_1 , $\bar{D}_1 \subset D$, a harmonic function $\varphi(z)$, continuous in \bar{D}_1 , be given. Suppose that the maximum (minimum) of the function $\Phi(z) = \psi(z) + \varphi(z)$ is attained at a point $z_0 \in D_1$, and on the boundary of the domain D_1 the inequality $\Phi(z) < \Phi(z_0)$ ($\Phi(z) > \Phi(z_0)$) holds. Then

$$\psi(z_0) = 0, \quad \text{grad } \varphi(z_0) = 0. \quad (1)$$

Indeed, let $\psi(z_0) > 0$. Consider the set $\tilde{D}_1 = \{z : \psi(z) > 0, \Phi(z) = \Phi(z_0)\}$. It is not hard to verify, using harmo-

* In paper ⁽¹⁾ a proof of the assertion of the theorem was given under the additional assumption that $\psi(z)$ has summable second derivatives. The theorem formulated here strengthens the main result of paper ⁽¹⁾.

icity of $\Phi(z)$ in a neighborhood of the point $z \in D_1$, in which $\psi(z) \neq 0$, that the set \tilde{D}_1 is open and that on its boundary $\psi(z) = 0$. By virtue of the maximum principle proved above, $\psi(z) \equiv 0$ in \tilde{D}_1 , which contradicts the assumption $\psi(z_0) > 0$. From the same considerations it follows that $\psi(z_0)$ cannot be negative.

Suppose now that $\text{grad } \varphi(z_0) \neq 0$. In this case $\text{grad } \psi(z_0) \neq 0$, and the set of zeros of the function $\psi(z)$ is, in a neighborhood of the point z_0 , a smooth curve and, consequently, is removable, i.e., the function $\psi(z)$ is harmonic in some neighborhood of the point z_0 . The function $\Phi(z)$, harmonic in a neighborhood of the point z_0 , attains an extremum at z_0 . Consequently, $\Phi(z) \equiv \Phi(z_0)$ in this neighborhood. As was shown above, from $\Phi(z) = \Phi(z_0)$ it follows that $\psi(z) = 0$. Thus, $\varphi(z) \equiv \varphi(z_0)$ in a neighborhood of the point z_0 and, contrary to the assumption, $\text{grad } \varphi(z_0) \equiv 0$.

2°. The theorem is local in character, and it suffices to prove it for any disk D_1 lying in D . Denote by $\varphi(z)$ the function harmonic in the chosen disk D_1 and equal on the boundary of this disk to the function $\psi(z)$. Suppose that $\varphi \not\equiv \psi$ in D_1 . Then, without loss of generality, we may assume that the function

$$\Phi(z) = \varphi(z) - \psi(z)$$

attains a negative minimum at some point $z_0 \in D_1$. By virtue of (1), at the point z_0

$$\text{grad } \psi(z_0) = 0, \quad \psi(z_0) = 0.$$

From the isolatedness of the zeros of the gradient of the harmonic function $\varphi(z)$ it follows that the point z_0 is an isolated minimum of the function $\Phi(z)$. In other words, for sufficiently small ρ_0 , $\Phi(z) > \Phi(z_0)$ when $0 < |z - z_0| \leq \rho_0$.

Denote by m the order of the zero of the function $\varphi(z) - \varphi(z_0)$ at the point z_0 . In polar coordinates r, θ , with center at the point z_0 , for a suitable choice of the

initial direction of the angle θ we have:

$$\varphi(z) - \varphi(z_0) = cr^m \cos m\theta + O(r^{m+1}), \quad c > 0. \quad (2)$$

It follows from (2) that the level lines $\varphi(z) = \varphi(z_0)$ divide the disk $|z - z_0| < \rho$, of sufficiently small radius ρ , $0 \leq \rho_1 < \rho_0$, into $2m$ domains $D_i^+(\rho), D_i^-(\rho), i = 1, \dots, m$. In the domains $D_i^+(\rho)$ the inequality $\varphi(z) > \varphi(z_0)$ holds, and in the domains $D_i^-(\rho)$ the opposite inequality holds. By the choice of ρ , $\Phi(z) > \Phi(z_0)$ when $0 < |z - z_0| \leq \rho$. Consequently, $\varphi(z) - \psi(z) > \varphi(z_0)$ and $\psi(z) < 0$ for $z \in D_i^-(\rho), z \neq z_0$.

It will be proved below that for some ρ , $0 < \rho \leq \rho_1$, there is at least one domain $D_i^+(\rho)$ with the following two properties: a) $\psi(z)$ vanishes at some interior points of the domain $D_i^+(\rho)$; b) on the arc of the circle $|z - z_0| = \rho$ that is part of the boundary of the domain $D_i^+(\rho)$, the inequality $\psi(z) \leq 0$ holds. The totality of these two facts contradicts the maximum principle established in 1°, and this proves the theorem.

Introduce the function

$$\Phi_\varepsilon(z) = \varphi(z) - \varepsilon \log |z - z_0| - \psi(z), \quad \varepsilon > 0.$$

On the circle $|z - z_0| = \rho_1$ the inequality $\Phi(z) > \Phi(z_0)$ holds. There exists, therefore, an $\varepsilon_0 > 0$ such that for $\varepsilon < \varepsilon_0$ the minimum of $\Phi_\varepsilon(z)$ in the disk $|z - z_0| \leq \rho_1$ is attained only at interior points z_ε of this disk. It is clear that the radius of the minimal disk in which the points z_ε are located tends to zero as $\varepsilon \rightarrow 0$. As was shown in item 1°, the points z_ε are zeros of the function $\psi(z)$. The points $z_\varepsilon \in \overline{D_j^-(\rho_1)}$, $j = 1, \dots, m$, and, consequently, for some $i = i_0$ in the domain $D_{i_0}^+(\rho_1)$ there exists a sequence of zeros z_ε of the function $\psi(z)$, converging as $\varepsilon \rightarrow 0$ to the point z_0 . The domain $D_{i_0}^+(\rho), \rho \leq \rho_1$, will henceforth be denoted by $D^+(\rho)$.

At the points z_ε , by virtue of (1), the relation

$$\text{grad} [\varphi(z_\varepsilon) - \varepsilon \log |z_\varepsilon - z_0|] = 0$$

holds, or, in polar coordinates,

$$\frac{\partial \varphi(z_\varepsilon)}{\partial r} - \frac{\varepsilon}{r_\varepsilon}, \quad \frac{\partial \varphi(z_\varepsilon)}{\partial \theta} = 0. \quad (3)$$

Differentiating the asymptotic formula (2), we find that

$$\theta_\varepsilon = \frac{2k\pi}{m} + O(r_\varepsilon), \quad \frac{\partial^2 \varphi(z_\varepsilon)}{\partial \theta^2} = -cm^2 r_\varepsilon^m + O(r_\varepsilon^{m+1}). \quad (4)$$

It is easy to establish, using (3) and (4), that the maximum of the function $\varphi(z)$ on the arc of the circle $|z - z_0| = |z_\varepsilon - z_0|$, $z_\varepsilon \in D^+(\rho_1)$, lying in $D^+(\rho_1)$, is attained, for small $|z_\varepsilon - z_0|$, at the point z_ε . On the other hand, at the point z_ε we have a minimum of the function $\Phi_\varepsilon(z)$, and, consequently, on the circle $|z - z_0| = |z_\varepsilon - z_0|$ the inequality $\varphi(z) - \psi(z) \geq \varphi(z_\varepsilon)$ holds. Thus, on the arc of this circle lying in $D^+(\rho_1)$, $\psi(z) < 0$ for $z \neq z_\varepsilon$. As the domain $D_{i_0}^+(\rho)$, whose existence we had to prove, one may take the domain $D^+(|z_\varepsilon - z_0|)$, where $z_\varepsilon \in D^+(\rho_1)$ and ε is sufficiently small.

Theorem 1'. *Let in the domain D there be given a uniformly elliptic operator*

$$L\varphi = a_{11}\varphi_{xx} + 2a_{12}\varphi_{xy} + a_{22}\varphi_{yy} + b_1\varphi_x + b_2\varphi_y + c\varphi$$

with coefficients satisfying the Hölder condition. If $\psi(z) \in C^1$ in the domain D and is a regular solution of the equation $L\psi = 0$ in a neighborhood of each point $z \in D$ at which $\psi(z) \neq 0$, then $\psi(z)$ is a regular solution of the equation $L\psi = 0$ everywhere in the domain D .

Proof. It suffices to prove the assertion of the theorem for some neighborhood of each point $z \in D$. Let z_0 be a distinguished point. In a sufficiently small neighborhood of the point z_0 there exists a positive solution $\omega(z)$ of the equation $L\omega = 0$. The substitution $\psi(z) = \omega(z)\bar{\psi}(z)$ leads to an equation in which $c(z) \equiv 0$ in a neighborhood of the point z_0 . Thus, without restricting generality, we may assume (and this will be assumed below) that $c(z) \equiv 0$ in the domain D . In part 1° of the proof of Theorem 1 only the maximum principle is essentially used, and all assertions of that part are valid for elliptic equations with $c(z) \equiv 0$.

The arguments of part 2° are based on two facts: the asymptotic representation (2) and the isolation of the zeros of the gradient of a regular solution. It is known that solutions of elliptic equations of the type under consideration possess these two properties (see, for example, (2)).

In order to use the arguments of part 2° more directly, we transform the operator L in a neighborhood of the distinguished point $z_0 \in D$ by a change of independent variables to the form $L\varphi = \Delta\varphi + \vec{a} \text{ grad } \varphi$. Keeping the old notation for the new variables, we obtain that in a neighborhood of the point z_0 , a minimum of the function $\Phi(z) = \varphi(z) - \psi(z)$,

$$\varphi(z) - \varphi(z_0) - cr^m \cos m\theta + o(r^{m+\lambda}), \quad c > 0, \quad (5)$$

where λ is the Hölder exponent of the coefficients of the operator L . The asymptotic representation (5) admits double differentiation. The function $\Phi_\varepsilon(z)$ is defined in our case as follows:

$$\Phi_\varepsilon(z) = \varphi(z) - \varepsilon g(z) - \psi(z), \quad \varepsilon > 0,$$

where $g(z)$ is the fundamental solution of the equation $Lg = 0$ with singularity at the point z_0 . It is known that

$$g(z) = \log |z - z_0| + o(|z - z_0|^\lambda). \quad (6)$$

Formula (6) also admits twofold differentiation. Denote by $s\rho$ the length of the arc of the level line

$$g(z) = \log \rho; \quad (7)$$

the second of equalities (3) will then be written in the form $\partial\varphi(z_\varepsilon)/\partial s = 0$. The cosine of the angle between the normal to the level line (7) and the vector $z - z_0$ is $o(\rho^\lambda) + 1$, and the curvature \varkappa of the level line (7) admits the representation

$$\varkappa = \frac{1}{\rho} + o(\rho^{\lambda-1}).$$

This makes it possible to obtain relations analogous to (4):

$$\theta_\varepsilon = \frac{2k\pi}{m} + o(r_\varepsilon^\lambda), \quad \frac{\partial^2\varphi(z_\varepsilon)}{\partial s^2} = -cm^2r_\varepsilon^m + o(r_\varepsilon^{m+\lambda}).$$

The arguments of item 2°, after these remarks, carry over without essential changes to the case of equations of the form $\Delta\varphi + \vec{a} \text{ grad } \varphi = 0$. The role of the domains $D_i^+(\rho)$ is now played by domains bounded by the level lines $\varphi(z) = \varphi(z_0)$ and by an arc of the level line (7).

Remark. As M. M. Lavrent'ev pointed out to the author, from the removability of level sets there follows the uniqueness of the solution of the Cauchy problem with data on an arbitrary set γ satisfying the condition: the set γ divides some domain D into two domains D_1 and D_2 . Indeed, suppose that there exist two solutions $\varphi'(z)$, $\varphi''(z)$ of the Cauchy problem with data on γ and with domain of definition $\widetilde{D}_1 \subset D_1$. Define the function

$$\psi(z) = \begin{cases} 0, & z \in D_2, \\ \varphi'(z) - \varphi''(z), & z \in \widetilde{D}_1 \cup \gamma. \end{cases}$$

From the removability of the set γ for the function $\psi(z)$ it follows that $\psi(z) \equiv 0$ in $D_2 \cup \widetilde{D}_1 \cup \gamma$.

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Note: Figure translations are in progress. See original paper for figures.

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