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Abstract

Full Text

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On the Absolute Convergence of Fourier Series

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Let $f(x) \in L^2_{2\pi}$ be a 2π-periodic function. Put

$$ {}^s\Delta_t^p f(x) = \sum_{k=0}^{p} (-1)^k C_p^k f[x + (p-2k)t], $$

$$ L^{(p)}(h, x, f) = \frac{1}{h} \int_0^h {}^s\Delta_t^p f(x)\, dt, $$

$$ L_2^{(p)}(h, f) = \frac{1}{h} \left\{ \sup_{0 \le u \le h} \int_{-\pi}^{\pi} \left[\int_0^u {}^s\Delta_t^p f(x)\, dt \right]^2 dx \right\}^{1/2}, $$

$$ {}^s\omega_p(h, f) = \sup_{0 \le t \le h} \sup_{-\pi \le x \le \pi} \left| {}^s\Delta_t^p f(x) \right|, $$

$$ {}^s\omega_p^{(2)}(h, f) = \left\{ \sup_{0 \le t \le h} \int_{-\pi}^{\pi} \left[{}^s\Delta_t^p f(x) \right]^2 dx \right\}^{1/2}. $$

Lemma 1. Suppose

$$ f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). $$

Then

$$ \frac{1}{n^{2p}} \sum_{k=1}^{n} (a_k^2 + b_k^2) k^{2p} \le C(p) \int_{-\pi}^{\pi} \left[L^{(p)}\left(\frac{1}{n}, x, f\right) \right]^2 dx, $$

where p is any natural number, and $C(p)$ is a constant depending only on p.

Lemma 2. If $0 < m \le 2$, then

$$\sum_{k=2^{\gamma-1}+1}^{2^\gamma}(|a_k|^m+|b_k|^m)\le C_1(p)\left\{\int_{-\pi}^{\pi}\left[L^{(p)}\left(\frac{1}{2^\gamma},x,f\right)\right]^2dx\right\}^{m/2}(2^\gamma)^{1-m/2},$$

where p and γ are any natural numbers, and $C_1(p)$ is a constant depending only on p.

Lemma 3. If $2^{l-1}\le n<2^l$, then

$$\sum_{k=n}^{\infty}k^{\alpha-1}E\left(\frac{1}{k}\right)\ge C(\alpha)\sum_{\gamma=l+1}^{\infty}(2^\gamma)^\alpha E\left(\frac{1}{2^\gamma}\right),$$

where $E\left(\frac{1}{k}\right)\downarrow 0$ as $k\uparrow\infty$; α is any real number; $C(\alpha)$ is a constant depending only on α; n and l are natural numbers.

Theorem 1. If $0<m\leqslant 2$, then

$$\sum_{k=n}^{\infty}(|a_k|^m+|b_k|^m)\leqslant$$

$$\leqslant C(p,m)\left\{\sum_{k=n}^{\infty}\frac{\left[L_2^{(p)}\left(\frac{1}{k},f\right)\right]^m}{k^{m/2}}+\left[L_2^{(p)}\left(\frac{1}{n},f\right)\right]^m n^{1-m/2}\right\},$$

where $C(p,m)<+\infty$ is a constant depending only on p and m.

Proof. Let $2^{l-1}\leqslant n<2^l$. Then

$$\sum_{k=n}^{\infty}(|a_k|^m+|b_k|^m)\leqslant\sum_{k=n}^{2^l}(|a_k|^m+|b_k|^m)+\sum_{\gamma=l+1}^{\infty}\sum_{k=2^{\gamma-1}+1}^{2^\gamma}(|a_k|^m+|b_k|^m)=$$

$$=O\left(\sum_{k=n}^{2n}(|a_k|^m+|b_k|^m)+\sum_{\gamma=l+1}^{\infty}\left\{\int_{-\pi}^{\pi}\left[L^{(p)}\left(\frac{1}{2^\gamma},x,f\right)\right]^2dx\right\}^{m/2}(2^\gamma)^{1-m/2}\right)=I_1+I_2.$$

Let us estimate I_1:

$$\sum_{k=n}^{2n}(|a_k|^m+|b_k|^m)\leqslant\left\{\sum_{k=n}^{2n}(a_k^2+b_k^2)\right\}^{m/2}n^{1-m/2}\leqslant$$

$$\leqslant C_2(p)\left\{\int_{-\pi}^{\pi}\left[L^{(p)}\left(\frac{1}{2n},x,f\right)\right]^2 dx\right\}^{m/2} n^{1-m/2} \leqslant C_3(p)\left[L_2^{(p)}\left(\frac{1}{n},f\right)\right]^m n^{1-m/2}.$$

Passing to the estimate of I_2, we have

$$\sum_{\gamma=l+1}^{\infty}\left\{\int_{-\pi}^{\pi}\left[L^{(p)}\left(\frac{1}{2^\gamma},x,f\right)\right]^2 dx\right\}^{m/2}(2^\gamma)^{1-m/2} \leqslant \sum_{\gamma=l+1}^{\infty}\left[L_2^{(p)}\left(\frac{1}{2^\gamma},f\right)\right]^m (2^\gamma)^{1-m/2}.$$

Applying* Lemma 3, we obtain

$$\sum_{\gamma=l+1}^{\infty}\left[L_2^{(p)}\left(\frac{1}{2^\gamma},f\right)\right]^m (2^\gamma)^{1-m/2} \leqslant C(m)\sum_{k=n}^{\infty}\frac{\left[L_2^{(p)}\left(\frac{1}{k},f\right)\right]^m}{k^{m/2}}.$$

The rest is clear.

Corollary 1. It is not hard to verify that

$$L_2^{(p)}(h,f) \leqslant {}^s\omega_p^{(2)}(h,f).$$

Consequently, for $0 < m \leqslant 2$,

$$\sum_{k=n}^{\infty}(|a_k|^m+|b_k|^m) = O\left(\sum_{k=n}^{\infty}\frac{\left[{}^s\omega_p^2\left(\frac{1}{k},f\right)\right]^m}{k^{m/2}}\right)+\left[{}^s\omega_p^{(2)}\left(\frac{1}{n},f\right)\right]^m n^{1-m/2}.$$

In particular, if $f(x)$ has l derivatives, $f^{(l)}(x) \in \operatorname{Lip}\alpha$, and $m(l+\alpha)+m/2>1$, then

$$\sum_{k=n}^{\infty}(|a_k|^m+|b_k|^m) = O\left(n^{1-m(l+\alpha+1/2)}\right).$$

For $l = 0$ this is a result of Lorentz ([1]).

$$* \text{ For } E\left(\frac{1}{k}\right) \text{ we take } \left\{\sup_{0\leqslant u\leqslant 1/k}\int_{-\pi}^{\pi}\left[\int_0^u {}^s\Delta_t^p f(x)\,dt\right]^2 dx\right\}^{1/2}.$$

Corollary 2. If

$$\sup_{0<t\leq 1/n}\int_{-\pi}^{\pi}|\,{}^s\Delta_t^p f(x)|\,dx = O\left(\frac{1}{n}\right), \qquad \sum_{k=1}^{\infty}\frac{\left[{}^s\omega_p\left(\frac{1}{k},f\right)\right]^{1/2}}{k} < +\infty,$$

then

$$\sum_{k=1}^{\infty}(|a_k|+|b_k|) < +\infty. \tag{1}$$

From this, in particular, the following two theorems follow:

Theorem A* (A. Zygmund ([2])). *If $f(x)$ has bounded variation and*

$$\sum_{k=1}^{\infty}\frac{\sqrt{\omega\left(\frac{1}{k},f\right)}}{k} < +\infty,$$

where

$$\omega(\delta,f) = \sup_{|x_1-x_2|<\delta}|f(x_1)-f(x_2)|,$$

then relation (1) is satisfied.

Theorem B (F. I. Harshiladze ([4])). *If $f(x)$ is such that for all x and h*

$$|f(x+h)+f(x-h)-2f(x)| \leq Mh^{\alpha}$$

with $\alpha > 0$, and, moreover, $f(x)$ has bounded second variation on $[0, 2\pi]$, then its Fourier series converges absolutely.

Indeed, Theorem A is obvious, while Theorem B follows from the equality

$$\int_{-\pi}^{\pi}|f(x+t)-2f(x)+f(x-t)|\,dx = O(t), \tag{2}$$

which is valid whenever $f(x)$ has bounded second variation ([5]). Since ([6]) it does not follow from (2) that $f \in V_2$, it is clear that Corollary 2 is stronger than Theorems A and B.

Let $F(x)$ be an antiderivative of $f(x)$.

Theorem 1. *If*

$$\sum_{k=1}^{\infty} k^{1/2}\omega_2^{(2)}\left(F, \frac{1}{k}\right) < +\infty,$$

then relation (1) is satisfied.

Corollary 1. (Theorem of Sas ([7])). *If*

$$\sum_{k=1}^{\infty} \frac{\omega_1^{(2)}(1/k, f)}{\sqrt{k}} < +\infty,$$

then relation (1) is satisfied.

Corollary 2. From

$$\sum_{k=1}^{\infty} \left\{ k^{\,s}\omega_2\left(F, \frac{1}{k}\right) \sup_{0<t\leq 1/k} \int_{-\pi}^{\pi} |F(x+t) - 2F(x) + F(x-t)|\, dx \right\}^{1/2} < +\infty$$

it follows that (1) holds.

Corollary 3. *If*

$$\sum_{k=1}^{\infty} \sqrt{k}^{\,s}\omega_2\left(F, \frac{1}{k}\right) < +\infty,$$

then (1) holds.

* The assertion formulated in the text is due to Salem ([3]), who notes that it follows immediately from Zygmund’ s results.

Hence it follows

Theorem (S. N. Bernstein [8]). The relation

$$\sum_{k=1}^{\infty} \frac{\omega(1/k, f)}{\sqrt{k}} < +\infty$$

implies (1).

Corollary 4. If

$$\sup_{0\leq t\leq 1/n} \int_{-\pi}^{\pi} |F(x+t) - 2F(x) + F(x-t)|\, dx = O\left(\frac{1}{n^2}\right),$$

then from

$$\sum_{k=1}^{\infty} \frac{\sqrt[5]{\omega_2(F, 1/k)}}{\sqrt{k}} < +\infty$$

follows (1).

This assertion contains the theorem of A. Zygmund mentioned above.

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Note: Figure translations are in progress. See original paper for figures.