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ON THE THEORY OF SU_6-SYMMETRY OF ELEMENTARY PARTICLES

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Recently the question of relativistic SU_6-symmetry has been widely discussed. In order to obtain an S-matrix possessing such a symmetry, it is necessary to construct unitary representations of an inhomogeneous group $P(SU_6)$, for which the small group is SU_6. The construction of unitary representations of the group $P(SU_6)$ is the subject of the present paper.

The algebra of generators of the group $P(SU_6)$ was studied by the authors ([1,2]). It turned out that the derivation of the commutation relations has a general character and is readily generalized to the case of groups $P(SU_r)$ with SU_r as the small group. Bearing in mind that the comparison of relativistic SU_6-symmetry with the symmetries SU_r is also of interest, we shall carry out the calculations for arbitrary r. The Poincaré group is the special case $P(SU_r)$ for $r = 2$. The commutation relations in $P(SU_r)$ have the form

$$\begin{aligned} [M_a, M_b] &= iF_{abc}M_c, \quad [P_\alpha, M_b] = iF_{\alpha b\sigma}P_\sigma, \\ [N_a, M_b] &= iF_{abc}N_c, \quad [P_\alpha, N_b] = D_{\alpha b\sigma}P_\sigma, \\ [N_a, N_b] &= iF_{abc}M_c, \quad [P_\alpha, P_\beta] = 0. \end{aligned} \tag{1}$$

Here Latin indices run through the values $1, \dots, r^2 - 1$, and Greek indices $0, 1, \dots, r^2 - 1$. As before ([1,2]), the generators M_a describe "spatial" rotations, N_a generate Lorentz-like transformations, and P_α are displacement operators (r^2 components).

Introduce a complete system of r^2 complex Hermitian matrices Λ_α, having r rows and r columns, with the normalization condition $\mathrm{Sp}\,\Lambda_\alpha\Lambda_\beta = \nu\delta_{\alpha\beta}$, where $\Lambda_0 = (\nu/r)^{1/2}E$. Then the structure constants F and D can be found with the aid of the matrices Λ_α

$$[\Lambda_\alpha, \Lambda_\beta] = 2iF_{\alpha\beta\sigma}\Lambda_\sigma; \qquad \{\Lambda_\alpha, \Lambda_\beta\} = 2D_{\alpha\beta\sigma}\Lambda_\sigma;$$

$$F_{\alpha\beta\gamma} = -\frac{1}{2\nu}\,\mathrm{Sp}(\Lambda_\alpha[\Lambda_\beta, \Lambda_\sigma]); \qquad D_{\alpha\beta\gamma} = \frac{1}{2\nu}\,\mathrm{Sp}(\Lambda_\alpha\{\Lambda_\beta, \Lambda_\sigma\}). \tag{2}$$

The subgroup $LS(r)$, constructed on the generators M_a and N_a, is an analogue of the homogeneous Lorentz group. The representations of this group can be obtained in exactly the same way as the representations for the case $r = 6$ ([2]).

The key point of the theory of symmetry based on the group $P(SU_r)$ should be regarded as the construction of the explicit form of the generalized Wigner operator α ([3]). The operator α is defined as a transformation of the group $LS(r)$ which takes the standard momentum p^0 into p

$$\alpha_{\mu\nu}(p)p^0_\nu = p_\mu. \tag{3}$$

For representations with a positive-definite invariant form $\det \Lambda_\mu p_\mu = x^r$, the standard momentum may be chosen in the form $p^0_0 = x$, $p^0_a = 0$. Equation (3) determines $\alpha(p)$ up to a transformation R belonging to the small group, since $Rp^0 = p^0$. Making use of

thereby, we choose as α a pure Lorentz transformation generated by the generators N_a. If the momentum p is written by means of the matrix $\hat{p} = \Lambda_\alpha p_\alpha$, then

$$\hat{\alpha}(p)\hat{p}^0\hat{\alpha}(\hat{p}) = \hat{p}, \tag{3'}$$

where $\hat{\alpha}(p)$ is a square $r \times r$ matrix.

Consider the basis SU_r-vector n_a, for which the relation

$$(\Lambda_a n_a)_{ij} = \xi_i \xi^*_j - \frac{1}{r}\delta_{ij} \equiv \theta_{ij}, \qquad i, j = 1, 2, \dots, r, \tag{4}$$

holds, where ξ_j belongs to a set of r complex numbers $\xi_1, \dots, \xi_r$ with the normalization $\sum \xi^*_j \xi_j = 1$. The matrix $S_{ij} = \xi_i \xi^*_j$ has the property $S^m = S$. Therefore

$$\theta^m = f(m, r)\left(\theta + \frac{1}{r}\right) + \left(-\frac{1}{r}\right)^m, \qquad f(m, r) = \left(\frac{r-1}{r}\right)^m - \left(-\frac{1}{r}\right)^m. \tag{5}$$

These formulas are sufficient for calculating $\hat{\alpha}(p)$.

The simplest form of the Hermitian transformation $\hat{\alpha}(p)$ is

$$\hat{\alpha}(p) = \exp[(\Lambda_a n_a)\beta] = \exp(\theta\beta), \tag{6}$$

where the parameter β depends on p. From (4) and (5) it follows that

$$\hat{\alpha}(p) = \Phi(r,\beta)\left(\theta + \frac{1}{r}\right) - \exp\left(-\frac{1}{r}\beta\right),$$

$$\Phi(r,\beta) = \exp\frac{r-1}{r}\beta - \exp\left(-\frac{1}{r}\beta\right). \tag{7}$$

Substituting (7) into (3′), we obtain

$$\frac{\chi}{r}\Phi(2\beta) + \chi\exp\left(-\frac{2}{r}\beta\right) = \sqrt{\frac{\nu}{r}}\,p_0 \equiv \rho_0,$$

$$p_k = \rho n_k, \qquad \rho = \chi\Phi(2\beta). \tag{8}$$

Consequently,

$$\beta = -\frac{r}{2}\ln\frac{q}{\chi}, \qquad q = \rho_0 - \frac{\rho}{r}. \tag{9}$$

Formula (9) together with (7) gives the solution of the problem of the Wigner operator for the case of the group $P(S\hat{U}_r)$.

For the construction of irreducible unitary representations it is necessary to know the invariant $Z = \det \Lambda_\mu p_\mu$, characterizing momentum space. The expression for Z in the case $r = 6$ in terms of the $\hat{M}$-invariants composed of the momenta was obtained earlier [(1)] without using the properties of n_a. From the form of $\hat{\alpha}$ it is clear that in the theory an essential role belongs to the quantities

$$t = \rho_0 + \frac{r-1}{r}\rho, \qquad q = \rho_0 - \frac{\rho}{r}. \tag{10}$$

We shall seek Z as a function of q and t from the condition of invariance of Z with respect to Lorentz-like transformations generated by N_a. This means that Z commutes with N_a. When acting on a scalar function of the momentum, the generator N_a is equivalent to N_a^0

$$N_a^0 = -D_{\alpha\beta a}p_\beta\frac{\partial}{\partial p_\alpha}. \tag{11}$$

It is convenient to introduce $N = n_{aN}a$. Then from (10) and (11) we have

$$[N,q] = \frac{1}{r}q, \qquad [N,t] = -\frac{r-1}{r}\,t. \tag{12}$$

The determinant Z is expressed simply as

$$Z = q^{r-1} t. \tag{13}$$

Let us proceed to the study of the generators M_a and N_a. With the aid of (6) and (9) it can be shown that, in the unitary representation, the generators M_a and N_a contain orbital parts N_a^0 and M_a^0

$$M_a = M_a^0 + \mathfrak{M}_a; \qquad N_a = N_a^0 + K_{ab}\mathfrak{M}_b, \tag{14}$$

where $\mathfrak{M}_a$ is the generator of "spin" transformations; moreover

$$M_a^0 = -iF_{abc}p_b \frac{\partial}{\partial p_c}. \tag{15}$$

For K_{ab} one may write

$$K_{ab} = \frac{1}{\nu}\,\mathrm{Im}\,\mathrm{Sp}\left(\Lambda_b \hat{\alpha}^{-1}(p)[2N_a^0 + \Lambda_a, \hat{\alpha}(p)]\right) =$$

$$= 2\left[\frac{q}{\rho}\left(2 - b - \frac{1}{b}\right) + 1 - b\right] F_{acb}n_c, \qquad b = \left(\frac{q}{\chi}\right)^{1/2}. \tag{16}$$

Equation (14) is a generalization, to the case of the group $P(SU_r)$, of the relations for the Poincaré group ($r = 2$) that were obtained by Yu. M. Shirokov ([4]).

In the course of computing (16) it is convenient to use the following relations:

$$[N_a^0, \varphi(q)] = \frac{n_a \nu}{r-1}\, q\, \frac{\partial \varphi}{\partial q},$$

$$[N_a^0, \theta] = \frac{n_a \nu r}{r-1}\left[\frac{r-2}{r} + \frac{\rho^0}{\rho}\right]\theta - \left(\frac{q}{\rho} + \frac{1}{2}\right)\Lambda_a.$$

Having the Wigner operator, one can solve a number of the problems posed earlier ([1,2,5]). With the aid of this operator one can obtain equations of motion in various forms and additional conditions, and write out their solutions explicitly. This operator is also important for the investigation of Bargmann–Wigner-type equations of motion and for determining the covariant vector B_a, entering the invariant $B_{aP}a = \chi^2$ ([5]).

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REFERENCES

[1] Yu. V. Novozhilov, I. A. Terent' ev, DAN, 165, No. 3 (1965).
[2] Yu. V. Novozhilov, I. A. Terent' ev, Vestn. LGU, No. 10, 5 (1965).
[3] E. P. Wigner, Ann. Math., 40, 149 (1939).
[4] Yu. M. Shirokov, DAN, 94, 857 (1954); C. Fronsdal, Phys. Rev., 113, 1367 (1959).
[5] Yu. V. Novozhilov, Phys. Lett., 16, No. 3 (1965).

Note: Figure translations are in progress. See original paper for figures.