SovietRxiv

Soviet-era science, translated into English

ON THE CONVERGENCE OF LACUNARY TRIGONOMETRIC SERIES ON SETS

MATHEMATICS

1965

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196501.25930

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

UDC 517.512.2

MATHEMATICS

V. F. EMEL' YANOV

ON THE CONVERGENCE OF LACUNARY TRIGONOMETRIC SERIES ON SETS

(Presented by Academician A. N. Kolmogorov, 15 IV 1965)

Zygmund proved the following theorems:

Theorem 1. If the upper limit of indeterminacy of a lacunary trigonometric series is different from $+\infty$ at each point of some interval, then the series of the moduli of the coefficients converges.

Theorem 2. If a lacunary trigonometric series is summable on a set of positive measure by a method generated by a matrix in which each column forms a sequence converging to zero, and the sums of the numbers written in the rows form a sequence converging to one, then the series of the squares of the coefficients converges.

A. N. Kolmogorov proved that, if a Fourier series turns out to be lacunary, then it converges almost everywhere. In the present note we indicate sets by which one can replace the interval in Theorem 1 and the set of positive measure in Theorem 2 without violating the validity of the assertions made in them. S. B. Stechkin proved that the interval in Theorem 1 can be replaced by a set of the second category. Therefore we shall be interested only in sets of the first category and, first of all, perfect sets of measure zero.

Definition 1. Let $\omega(n) \downarrow 0$ as $n \to \infty$. A set $A \subseteq [0, 2\pi]$ is called an M-$\omega(n)$-set if there exists on $[0, 2\pi]$ a nondecreasing function $F(x)$, constant on every interval (α, β) contained in $[0, 2\pi]$ and not containing points of A, not identically constant on $[0, 2\pi]$, and such that

$$\int_0^{2\pi} e^{-inx}\, dF(x) = O(\omega(n)).$$

The interval $[\alpha, \beta] \subseteq [0, 2\pi]$ is an M-n^{-1}-set. D. E. Men' shov was the first to construct a perfect set of measure zero which is an M-$\ln^{-1/2} n$-set. Littlewood was the first to prove the existence, for $\varepsilon > 0$, of a set of measure zero which is an M-$n^{-\varepsilon}$-set. Examples of perfect sets of measure zero which are M-$n^{-\varepsilon}$

SovietRxiv

$(0 < \varepsilon < 1/2)$-sets can be found, for example, in Salem' s work ([5]). In the present work the following is used essentially.

Lemma 1. Let on $[N, +\infty]$ a function $\omega(t)$ be given, having, at $+\infty$, the following properties: 1) $\omega(t) \downarrow 0$; 2) $\omega^{-1}(t) = O(t^2)$; 3) for any constant $C > 0$ there exists a constant C_1 such that $\omega(Ct) \leq C_1\omega(t)$. If the upper limit of a sequence of functions continuous on $[0, 2\pi]$ is different from $+\infty$ on some M-$\omega(n)$-set, then there exists an M-$\omega(n)$-set on which the sequence is uniformly bounded above.

Theorem 3. Let $\omega(t)$ satisfy the conditions of Lemma 1, $n_k \uparrow \infty$, $\Delta n_k = n_k - n_{k-1} \uparrow \infty$,

$$\sum_{k=1}^{\infty} k\omega^2(\Delta n_k) < +\infty.$$

If

$$\overline{\lim_{N\to\infty}} \left| \sum_{k=1}^{N} \rho_k \cos(n_k x + \varepsilon_k) \right| \neq \infty$$

on an $M - \omega(n)$-set, then

$$\sum_{k=1}^{\infty} \rho_k^2 < \infty.$$

Theorem 4. Let $\omega(t)$ satisfy the conditions of Lemma 1, let the sequence $\{n_k\}$ satisfy the conditions of Theorem 3, and let the matrix $A = (a_{nk})$ be such that: 1) for every n there exists k_n such that $a_{nk} = 0$ for $k > k_n$; 2) for every fixed k,

$$\lim_{n\to\infty} a_{nk} = 0;$$

3)

$$\lim \sum_{k=1}^{k_n} a_{nk} = 1.$$

If

$$\sum_{k=1}^{\infty} \rho_k \cos(n_k x + \varepsilon_k)$$

SovietRxiv

is summable to a finite number at each point of an $M-\omega(n)$-set by the method generated by the matrix A, then

$$\sum_{k=1}^{\infty} \rho_k^2 < +\infty.$$

From Theorems 3 and 4, as a corollary, we obtain the following theorems.

Theorem 5. If the limits of indeterminacy of a lacunary trigonometric series are finite at each point of some $M - \ln^{-1-\delta} n$-$(\delta > 0)$ set, then it is a Fourier series.

Theorem 6. Let the matrix A satisfy the conditions of Theorem 4. If a lacunary trigonometric series is summable at each point of some $M - \ln^{-1-\delta} n$-$(\delta > 0)$ set by the method generated by the matrix A, then it is a Fourier series.

The following theorem is a generalization of Theorem 1.

Theorem 7. If the upper limit of indeterminacy of a lacunary trigonometric series is different from $+\infty$ on some $M - n^{-\delta}$-$(\delta > 0)$ set, then it is the Fourier series of a continuous function.

Definition 2. A set E is called a $Z[\lambda]$-set if, from the finiteness of the upper limit of indeterminacy on E of a lacunary trigonometric series with degree of lacunarity greater than λ, there follows the convergence of the series of the moduli of its coefficients.

Definition 3. A set E is called a Z-set if there exists λ for which E is a $Z[\lambda]$-set.

In the work of V. V. Nemytskii ([4]), the concept of the A-k-property for closed sets was introduced.

Theorem 8. If a set E has the A-k-property and $k > 1$, then it is a Z-set.

It follows from Theorem 8 that there exist sets of uniqueness for the trigonometric series which are Z-sets. One can construct sets of uniqueness for the trigonometric series which are $Z[\lambda]$-sets for any $\lambda > 5$. But it is not known whether there exist sets of uniqueness for the trigonometric series which are $Z[1]$-sets.

Saratov State University
named after N. G. Chernyshevsky

Received
15 IV 1965

REFERENCES

1. N. K. Bari, *Trigonometric Series*, Moscow, 1961.

SovietRxiv

2. N. Wiener, A. Wintner, *Am. J. Math.* (Baltimore), **60**, 513 (1938).

3. O. S. Ivashev-Musatov, *Izv. Acad. Sci. USSR, Ser. Math.*, **20**, 179 (1956).

4. V. V. Nemytskii, *Mat. Sb.*, **33**, 5 (1926).

5. R. Salem, *J. Math. and Phys.* (Massachusetts Inst. Technology), **21**, 69 (1942).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.