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Abstract

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MATHEMATICS

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FLUCTUATIONS IN THE DISTRIBUTION OF FRACTIONAL PARTS

(Presented by Academician I. M. Vinogradov, 18 XI 1964)

Let on the half-interval $[0, 1)$, which we shall regard as a circle of unit length with a given initial point and direction of traversal, k numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ be given. Let a natural number m be given. If Δ is some half-interval lying on Δ^*, of length $1/2m$, then denote by $B_m(\Delta)$ the number of numbers among $\alpha_1, \alpha_2, \dots, \alpha_k$ belonging to Δ. Further, if Δ and Δ' are two half-intervals such that

$$\operatorname{mes}\Delta + \operatorname{mes}\Delta' = 1/2m,$$

then denote by $B_m(\Delta, \Delta')$ the number of numbers $\alpha_1, \alpha_2, \dots, \alpha_k$ belonging to $\Delta \cup \Delta'$.

For the study of additive problems, G. A. Freiman ([1]) established the following result:

Theorem. Suppose that for every half-interval Δ (of length $1/2$)

$$B_1(\Delta) \leqslant f(k).$$

Then

$$\left|\sum_{j=1}^{k} e^{2\pi i \alpha_j}\right| \leqslant 2f(k) - k.$$

We generalize this theorem.

Theorem 1. Let m be a given natural number. Suppose that the set

$$\alpha_1,\ \alpha_2,\ \dots,\ \alpha_k$$

is such that for every half-interval Δ (of length $1/2m$) and for every pair of half-intervals Δ and Δ' (of total length $1/2m$)

$$B_m(\Delta) \leqslant f(k), \qquad B_m(\Delta, \Delta') \leqslant f(k).$$

Then

$$\left|\sum_{j=1}^{k} e^{2\pi i m \alpha_j}\right| \leqslant 2mf(k) - k.$$

Proof. Since for any real γ

$$\left|\sum_{j=1}^{k} e^{2\pi i m(\alpha_j+\gamma)}\right| = \left|\sum_{j=1}^{k} e^{2\pi i m \alpha_j}\right|$$

and the sequence $\{\alpha_j + \gamma\}$ also has the property that for it $B_m(\Delta) \leqslant f(k)$ and $B_m(\Delta, \Delta') \leqslant f(k)$, we may assume that

$$\left|\sum_{j=1}^{k} e^{2\pi i m \alpha_j}\right| = \sum_{j=1}^{k} e^{2\pi i m \alpha_j} = \sum_{j=1}^{k} \cos 2\pi m \alpha_j,$$

$$\sum_{j=1}^{k} \cos 2\pi m \alpha_j = \sum_{r=0}^{m-1} \sum_{r/m \le \alpha_j < (r+1)/m} \cos 2\pi m \alpha_j =$$

* We shall assume that the right endpoint (in the direction of traversal) of Δ belongs to Δ, while the subsequent endpoint does not belong to Δ.

$$= \sum_{r=0}^{m-1} \Bigg(\sum_{r/m \le \alpha_j < r/m+1/4m} \cos 2\pi m \left(\alpha_j - \frac{r}{m}\right) -$$

$$- \sum_{r/m+1/4m \le \alpha_j < r/m+1/2m} \cos 2\pi m \left(\frac{r}{m} + \frac{1}{2m} - \alpha_j\right) -$$

$$- \sum_{r/m+1/2m \le \alpha_j < r/m+3/4m} \cos 2\pi m \left(\alpha_j - \frac{r}{m} - \frac{1}{2m}\right) +$$

$$+ \sum_{r/m+3/4m \le \alpha_j < (r+1)/m} \cos 2\pi m \left(\frac{r}{m} + \frac{1}{m} - \alpha_j\right)\Bigg) =$$

$$= 2\pi m \sum_{r=0}^{m-1} \left(\sum_{r/m \le \alpha_j < r/m+1/4m} \int_{\alpha_j - r/m}^{1/4m} \sin 2\pi mt\, dt - \right.$$

$$- \sum_{r/m+1/4m \le \alpha_j < r/m+1/2m} \int_{r/m+1/2m-\alpha_j}^{1/4m} \sin 2\pi mt\, dt -$$

$$- \sum_{r/m+1/2m \le \alpha_j < r/m+3/4m} \int_{\alpha_j - r/m - 1/2m}^{1/4m} \sin 2\pi mt\, dt +$$

$$\left. + \sum_{r/m+3/4m \le \alpha_j < (r+1)/m} \int_{r/m+1/m-\alpha_j}^{1/4m} \sin 2\pi mt\, dt \right) =$$

$$= 2\pi m \sum_{r=0}^{m-1} \left(\int_0^{1/4m} \left(\sum_{r/m \le \alpha_j < r/m+t} 1 - \sum_{r/m+1/2m-t \le \alpha_j < r/m+1/2m} 1 - \right. \right.$$

$$\left. - \sum_{r/m+1/2m \le \alpha_j < r/m+1/2m+t} 1 + \sum_{r/m+1/m-t \le \alpha_j < r/m+1/m} 1 \right) \sin 2\pi mt\, dt,$$

since

$$k = 2\pi m \int_0^{1/4m} \left(\sum_{0 \le \alpha_j < 1} 1 \right) \sin 2\pi mt\, dt =$$

$$= 2\pi m \sum_{r=0}^{m-1} \int_0^{1/4m} \left(\sum_{r/m \le \alpha_j < (r+1)/m} 1 \right) \sin 2\pi mt\, dt.$$

Adding, we obtain

$$\sum_{j=1}^{k} \cos 2\pi m \alpha_j + k = 2\pi m \sum_{r=0}^{m-1} \left(\int_0^{1/4m} \left(\sum_{(r+1)/m - t \le \alpha_j < r/m+1/m+t} 1 \right) \sin 2\pi mt\, dt + \right.$$

$$\left. + \int_0^{1/4m} \left(\sum_{\substack{r/m+1/2m-t \le \alpha_j < (r+1)/m \\ r/m \le \alpha_j < t}} 1 \right) \sin 2\pi mt\, dt \right) \le$$

$$\le 2\pi m \sum_{r=0}^{m-1} \left(f(k) \int_0^{1/4m} \sin 2\pi mt\, dt + f(k) \int_0^{1/4m} \sin 2\pi mt\, dt \right) = 2m f(k),$$

which was required to be proved.

This theorem makes it possible to obtain information on the presence of irregularities in the distribution of fractional parts, provided we are able to estimate the corresponding trigonometric sums from below.

We shall give two examples of applications of the theorem.

Example 1. Let $p > 2$ be a prime number, $(a, p) = 1$. As is known,

$$\left|\sum_{x=0}^{p-1} e^{2\pi i a x^2/p}\right| = \sqrt{p}.$$

From G. A. Freiman' s theorem and this estimate there immediately follows

Corollary 1. *For any $\varepsilon > 0$, there is a half-interval of length $1/2$ into which fall more than*

$$\frac{1}{2}p + \left(\frac{1}{2} - \varepsilon\right)\sqrt{p}$$

of the points $\{ax^2/p\}$, $x = 0, 1, \dots, p-1$.

From our Theorem 1 there follows

Corollary 2. *Let $p - 1 \geq m \geq 2$ be natural numbers. For any $\varepsilon > 0$, there is a half-interval Δ of length $1/2m$, or two half-intervals Δ, Δ' of the same (total) length, such that into them fall more than*

$$(p + \sqrt{p} - \varepsilon)/2m$$

of the points $\{ax^2/p\}$, $x = 0, 1, \dots, p-1$.

I. M. Vinogradov ([2]), p. 90, problem 12 §§, Ch. 5, proved that the number of fractional parts $\{ax^2/p\}$, $x = 0, 1, \dots, p-1$, falling into a half-interval of length $1/2m$, is approximately equal to

$$p/2m + \theta\sqrt{p}\ln p, \qquad |\theta| \leq 1$$

(and for two half-intervals of total length $1/2m$ this number is approximately equal to

$$p/2m + 2\theta\sqrt{p}\ln p).$$

It follows from the indicated corollaries that, for half-intervals of length $1/2$, even if Vinogradov' s formula can be sharpened, one can at most omit the factor $\ln p$, while for intervals of length

$$1/2m \approx \ln p/\sqrt{p}$$

Vinogradov' s formula cannot be improved in order of magnitude.

Example 2. Let $g \geq 2$ be a fixed natural number, and let p be an increasing natural number. We also prescribe a natural number m. By E_p we denote the set of points of the half-interval $[0, 1)$ for which there exists a half-interval of length $1/2m$, or two half-intervals Δ and Δ' (of total length $1/2m$), into the union of which fall no fewer than

$$p/2m + (\lambda/2m)\sqrt{p}$$

of the points $\{\alpha g^x\}$, $x = 0, 1, \dots, p-1$.

Corollary. *As $p \to \infty$ and for fixed real λ,*

$$\operatorname{mes} E_p \geq e^{-\lambda^2} + o(1).$$

Proof. We proceed from the result of M. P. Mineev ([3]): for fixed real λ,

$$\operatorname{mes} E \left\{ \alpha : 0 \leq \alpha < 1, \left| \sum_{x=0}^{p-1} e^{2\pi i m \alpha g^x} \right| > \lambda \sqrt{p} \right\} = e^{-\lambda^2} + o(1).$$

If $\alpha \notin E_p$, then for this α, by Theorem 1,

$$\left| \sum_{x=0}^{p-1} e^{2\pi i m \alpha g^x} \right| \leq 2m \left(\frac{p}{2m} + \frac{\lambda}{2m} \sqrt{p} \right) - p = \lambda \sqrt{p}.$$

Therefore, if

$$\alpha \in E \left(\left| \sum_{x=0}^{p-1} e^{2\pi i m \alpha g^x} \right| > \lambda \sqrt{p} \right),$$

then $\alpha \in E_p$, whence

$$\operatorname{mes} E_p \geq \operatorname{mes} E \left(\left| \sum_{x=0}^{p-1} e^{2\pi i m \alpha g^x} \right| > \lambda \sqrt{p} \right) = e^{-\lambda^2} + o(1),$$

as was required to prove.

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[1] G. A. Freiman, *Izv. Vyssh. Uchebn. Zaved.*, No. 6, 131 (1962).
[2] I. M. Vinogradov, *Foundations of Number Theory*, Moscow, 1949.
[3] M. P. Mineev, *UMN*, 14, issue 3, 169 (1959).

Note: Figure translations are in progress. See original paper for figures.