ON THE STRUCTURE OF ISOMETRIC TRANSFORMATIONS OF A SYMPLECTIC AND ORTHOGONAL VECTOR SPACE

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Abstract

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ON THE STRUCTURE OF ISOMETRIC TRANSFORMATIONS OF A SYMPLECTIC AND ORTHOGONAL VECTOR SPACE

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We consider an isometric linear transformation $\mathcal{A}$ of a nondegenerate bilinear-metric ([1], pp.245–267) vector space V with an orthogonal (symmetric) or symplectic (skew-symmetric) metric over an arbitrary field K of characteristic $\neq 2$. The author' s aim is to clarify the most general properties of isometric transformations of the space V, independent of the special features of the field, and to single out, if possible, the class of such isometric transformations that are described by these properties up to isomorphism.

This formulation of the problem has much in common with the classical investigations of the structure of orthogonal transformations of Euclidean spaces or unitary transformations of unitary spaces ([1]); on the other hand, it follows naturally from the results of the works ([3–5]).

In order to formulate the result, let us introduce several definitions. From the classical theory of elementary divisors it is known that the space V decomposes into a direct sum of subspaces invariant with respect to $\mathcal{A}$, corresponding to the decomposition of the characteristic polynomial of the transformation $\mathcal{A}$ into elementary divisors, which are powers of certain polynomials irreducible over the field K.

In the present case it is expedient to divide all elementary divisors into three types.

The first type will include elementary divisors of the form $p^n(\lambda)$, where $p(\lambda)$ is a polynomial irreducible over K of degree ≥ 2, and $p^n(\lambda)$, together with the root β, has the root $1/\beta$.

In the second type are grouped elementary divisors of the form $(\lambda \pm 1)^n$. All remaining elementary divisors belong to the third type. If the polynomial

$$q(\lambda) = \lambda^r + a_1\lambda^{r-1} + \cdots + a_r$$

belongs to the third type, then by the symbol $q(1/\lambda)$ we shall denote the polynomial having roots reciprocal to the roots of the polynomial $q(\lambda)$, and satisfying the relation

$$a_r q(1/\lambda) = a_r\lambda^r + a_{r-1}\lambda^{r-1} + \cdots + a_1\lambda + 1.$$

Taking into account the definitions given above, the space V can be written as the direct sum:

$$V = U_1(p_1, n_1) \oplus U_2(p_2, n_2) \oplus \cdots \oplus U_k(p_k, n_k) \oplus U_1(q_1, m_1) \oplus \cdots$$

$$\cdots \oplus U_l(q_l, m_l) \oplus U(\lambda \pm 1)^{r_1} \oplus \cdots \oplus U(\lambda \pm 1)^{r_s}.$$

Here the subspaces $U_i(p_i, n_i)$ correspond to the elementary divisors of the first type $p_i^{n_i}(\lambda)$, the subspaces $U_i(q_i, m_i)$ correspond to the elementary divisors of the third type $q_i^{m_i}(\lambda)$, and the subspaces $U(\lambda \pm 1)^{r_i}$ to the elementary divisors $(\lambda \pm 1)^{r_i}$.

Main assertion. For any isometric transformation $\mathcal{A}$, the space V decomposes into an orthogonal sum of nondegenerate subspaces invariant with respect to $\mathcal{A}$, of the types

$$U(p, n), \quad U(\lambda \pm 1)^{2k+1-\nu}, \quad U(q(\lambda), m) \oplus U(q(1/\lambda), m),$$

$$U_1(\lambda \pm 1)^{2k+\nu} \oplus U_2(\lambda \pm 1)^{2k+\nu},$$

where the subspaces

$$U(q(\lambda), m), \quad U(q(1/\lambda), m), \quad U_1(\lambda \pm 1)^{2k+\nu}, \quad U_2(\lambda \pm 1)^{2k+\nu}$$

are completely expressible (isotropic) ([1], [2]), $\nu = 1$ for a symplectic metric, $\nu = 0$ for an orthogonal one.

If, for some isometric transformation $\mathcal{A}$, in the orthogonal decomposition of the space V there occur only subspaces of the form

$$U(q(\lambda), m) \oplus U(q(1/\lambda), m)$$

and

$$U_1(\lambda \pm 1)^{2k+\nu} \oplus U_2(\lambda \pm 1)^{2k+\nu},$$

then the following is valid.

Theorem. Let V be a nondegenerate bilinear-metric space with an orthogonal or symplectic metric over an arbitrary field of characteristic $\neq 2$, and let $\mathcal{A}$ be an isometric transformation of this space with characteristic polynomial

$$f(\lambda) = q_1^{m_1}(\lambda) \cdots q_l^{m_l}(\lambda) \cdot (\lambda \pm 1)^{2k_1+\nu} \cdots (\lambda \pm 1)^{2k_s+\nu},$$

where $q_i^{m_i}(\lambda)$ are elementary divisors of the third type; then in the space V there is a basis such that its Gram matrix is equal to

$$\begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_{(i+s)/2} \end{pmatrix}, \qquad \text{where} \quad \Gamma_i = \begin{pmatrix} O & E \\ (-1)^\nu E & O \end{pmatrix},$$

and the matrix of the transformation $\mathcal{A}$ takes the form

$$\begin{pmatrix} B_1 & & & \\ & B_2 & & \\ & & \ddots & \\ & & & B_{(i+s)/2} \end{pmatrix},$$

where the matrices B_i correspond to the characteristic polynomials

$$f_i(\lambda) = f_{i1}(\lambda) \cdot f_{i1}^{m_i}(1/\lambda)$$

of the type

$$q_i^{m_i}(\lambda) \cdot q_i^{m_i}(1/\lambda)$$

or

$$(\lambda \pm 1)^{2k_i+\nu} \cdot (\lambda \pm 1)^{2k_i+\nu},$$

and

$$B_i = \begin{pmatrix} A_i & O \\ O & A_i^{-1} \end{pmatrix}, \qquad \text{with} \quad A_i = \begin{pmatrix} 0 & \cdots & 0 & -\alpha_r \\ 1 & \ddots & \vdots & \vdots \\ & \ddots & 0 & -\alpha_2 \\ & & 1 & -\alpha_1 \end{pmatrix},$$

if

$$f_{i1}(\lambda) = \lambda^r + \alpha_1 \lambda^{r-1} + \cdots + \alpha_r.$$

Remark. For the matrices A_i, if

$$f_{i1}(\lambda) = (\lambda \pm 1)^{2k_i+\nu},$$

one may propose one more canonical form:

$$A_i = \begin{pmatrix} \mp 1 & & & & \\ 1 & \mp 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & \mp 1 \end{pmatrix}.$$

The investigations were carried out by the methods of geometric algebra developed in the monograph ([2]).

The structure of isometric transformations in subspaces of the type

$$U(p, n)$$

and

$$U(\lambda \pm 1)^{2k+1-\nu}$$

depends essentially on the special features of the particular field K, and therefore is not considered in the present note.

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[4] Yu. B. Ermolaev, DAN, 132, No. 2, 257 (1960).
[5] Yu. B. Ermolaev, *Final Scientific Conference of Kazan State University for 1962*, Kazan, 1963, p. 25.

Note: Figure translations are in progress. See original paper for figures.