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Abstract

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ESTIMATES OF RATIONAL TRIGONOMETRIC SUMS OF A SPECIAL FORM

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An estimate of A. Weil (2) for a complete trigonometric sum with a polynomial with integral coefficients, p prime, $n < p$, $a_n \not\equiv 0 \pmod p$,

$$\left|\sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x}{p}\right]\right| \le (n-1)\sqrt{p}$$

becomes trivial when $n - 1 > \sqrt{p}$. In the present note we obtain estimates of rational sums with a polynomial of degree n of the form

$$S = \sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{Ax^n + Bx}{p}\right], \qquad S = \sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{Ax^n + Bx^k}{p}\right],$$

which are nontrivial for $n > \sqrt{p}$.

Theorem 1. Let $p \ge 3$ be prime; A, B, n natural numbers; $p > n$; $\delta = (n, p-1)$; $(A, p) = 1$; $(B, p) = 1$. The inequality

$$\left|\sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{Ax^n + Bx}{p}\right]\right| \le \frac{p}{\sqrt{\delta}}$$

holds.

Proof. Denote by Y the set of all elements y of the reduced residue system modulo p for which $y^n \equiv 1 \pmod p$. The set Y contains δ elements. For each fixed $y \in Y$, if x runs through a complete residue system modulo p, then xy also runs through a complete residue system modulo p. Consequently,

$$S = \sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{Ax^n + Bx}{p}\right] = \sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{A(xy)^n + Bxy}{p}\right] =$$

$$= \sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{Ax^n + Bxy}{p}\right] = S_y.$$

Summing over all $y \in Y$, we obtain

$$\delta S = \sum_{y\in Y} \sum_{x=0}^{p-1} \exp\left[2\pi i \, \frac{Ax^n + Bxy}{p}\right].$$

Squaring the inequality

$$\delta |S| \leq \sum_{x=0}^{p-1} \left| \sum_{y\in Y} \exp\left[2\pi i \, \frac{Bxy}{p}\right] \right|$$

and applying the Cauchy–Bunyakovsky inequality, we obtain

$$\delta^2 |S|^2 \leq \sum_{x=0}^{p-1} \left| \sum_{y\in Y} \exp\left[2\pi i \, \frac{Bxy}{p}\right] \right|^2 =$$

$$= p \sum_{y_1\in Y} \sum_{y_2\in Y} \sum_{x=0}^{p-1} \exp\left[2\pi i \, \frac{B(y_1 - y_2)x}{p}\right] = p^2 \sum_{\substack{y_1 \equiv y_2 \pmod{p} \\ y_1, y_2 \in Y}} 1 = p^2 \delta,$$

i.e.

$$|S| \leq p/\sqrt{\delta}.$$

Corollary. Let p be a prime number, $n \mid p-1$, $(A,p) = 1$, $(B,p) = 1$. Then

$$\left| \sum_{x=0}^{p-1} \exp\left[2\pi i \, \frac{Ax^n + Bx}{p}\right] \right| \ll p^{5/6}.$$

Proof. Multiplying the squared estimate of Theorem 1 by A. Weil' s estimate

$$\left| \sum_{x=0}^{p-1} \exp\left[2\pi i \, \frac{Ax^n + Bx}{p}\right] \right| \leq (n-1)\sqrt{p},$$

we obtain

$$|S|^2 |S| \ll \frac{p^2}{n} \, n p^{1/2} = p^{5/2}$$

or

$$|S| \ll p^{5/6}.$$

When the two estimates are multiplied, they lose part of their strength; however, since the left-hand side of the resulting upper bound does not depend on n, the result has an attractive form.

Denote by $B(T)$ the number of solutions of the congruence $Ax^n + Bx \equiv y \pmod{p}$ in the numbers $0 \le x \le p-1,\ 0 \le y \le T-1$.

Theorem 2. Let $(A,p)=1,\ (B,p)=1,\ n \mid p-1$.
Then

$$B(T) = T + \theta p^{5/6} \ln p, \qquad |\theta| \le 1.$$

Proof. Representing the number $B(T)$ in the form

$$B(T) = \sum_{y=0}^{T-1} \sum_{x=0}^{p-1} \frac{1}{p} \sum_{a=0}^{p-1} \exp\left[2\pi i a \frac{Ax^n + Bx - y}{p}\right]$$

and separating the terms with $a=0$, we obtain

$$B(T) = T + \frac{1}{p} \sum_{a=1}^{p-1} \left(\sum_{y=0}^{T-1} \exp\left[-2\pi i \frac{ay}{p}\right] \right) \left(\sum_{x=0}^{p-1} \exp\left[2\pi i \frac{Ax^n + Bx}{p}\right] \right).$$

From the known estimate

$$\left| \sum_{a=1}^{p-1} \left(\sum_{y=0}^{T-1} \exp\left[-2\pi i \frac{ay}{p}\right] \right) \right| < p \ln p$$

and the estimate of the corollary to Theorem 1, the assertion of Theorem 2 follows.

Theorem 3. Let p be a prime number; $a,\ b,\ k,\ n$ natural numbers; $n \mid p-1$; $(n,k)=1$; $\delta = (k, p-1)$; $(a,p)=1$; $(b,p)=1$.
Then

$$\left| \sum_{x=0}^{p-1} \exp\left[2\pi i \frac{ax^n + bx^k}{p}\right] \right| \ll \frac{p}{\sqrt{n}} + \sqrt{\delta - 1}\, p^{3/4}.$$

Proof. Choose the set Y to be the same as in Theorem 1. Similarly to the proof of Theorem 1, we have

$$\left|\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{ax^n+bx^k}{p}\right]\right|^2 \le \frac{p}{n^2}\sum_{y_1\in Y}\sum_{y_2\in Y}\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{bx^k\left(y_1^k-y_2^k\right)}{p}\right] =$$

$$= \frac{p^2}{n}+\frac{p}{n^2}\sum_{\substack{y_1,y_2\in Y\\ y_1\ne y_2}}\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{b\left(y_1^k-y_2^k\right)}{p}x^k\right].$$

Since, by the conditions of the theorem, $y_1^k \not\equiv y_2^k \pmod p$ for $y_1 \ne y_2$, the estimate of I. M. Vinogradov is applicable to the sum

$$\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{b\left(y_1^k-y_2^k\right)x^k}{p}\right]$$

$$\left|\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{b\left(y_1^k-y_2^k\right)}{p}x^k\right]\right| \le (\delta-1)\sqrt{p}.$$

Consequently,

$$\left|\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{ax^n+bx^k}{p}\right]\right|^2 \le \frac{p^2}{n}+\frac{p}{n^2}\,n(n-1)(\delta-1)\sqrt{p}.$$

From the inequality $\sqrt{a'+b'} \le \sqrt{a'}+\sqrt{b'}$ (for $a' \ge 0,\ b' \ge 0$) Theorem 3 follows.

In the case $(k,p-1)=1$ we obtain the estimate

$$\left|\sum_{x=0}^{p-1}\exp\left[2\pi i\,\frac{ax^n+bx^k}{p}\right]\right| \le \frac{p}{\sqrt{n}}.$$

This estimate is stronger than A. Weil’ s estimate for $n > p^{1/3}$. The estimate of Theorem 3 is stronger than A. Weil’ s estimate for

$$n > \max\left((1/2p)^{1/3},\ (1/2p)^{1/4}\sqrt{\delta-1}\right).$$

The method used in proving the estimates can be applied to deriving a variety of other inequalities. For example, the following assertion is true.

Theorem 4. Let $(a,p)=(b,p)=(c,p)=1;\ n_1 \mid p-1;\ n_2 \mid p-1;\ (n_1,n_2)=1;\ (k,p-1)=1;\ (k,n_1)=1.$

Then

$$\left|\sum_{x=0}^{p-1} \exp\left[2\pi i\,\frac{ax^{n_1}+bx^{n_2}+cx^k}{p}\right]\right| \le \sqrt{2}\,\frac{p}{\sqrt[4]{n_2}}.$$

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2. A. Weil, *Proc. Nat. Acad. Sci. U.S.A.*, **34**, 204 (1948).

Note: Figure translations are in progress. See original paper for figures.