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Abstract

Full Text

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PLANE POTENTIAL FLOWS WITH STATIONARY STREAMLINES

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Hydromechanics

§ 1. Consider the equations of plane potential motion of a polytropic gas

$$\frac{\partial u_i}{\partial t} + \frac{\partial \Delta}{\partial x_i} = 0, \qquad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_2}{\partial x_1}; \tag{1,1}$$

$$\frac{\partial \Delta}{\partial t} + 2u_k \frac{\partial \Delta}{\partial x_k} - u_i u_k \frac{\partial u_i}{\partial x_k} + c^2 \frac{\partial u_k}{\partial x_k} = 0; \tag{1,2}$$

$$c^2 = \chi\left(\Delta - \frac{u_k^2}{2}\right), \qquad i, k = 1, 2. \tag{1,3}$$

Here x_i are Cartesian coordinates, t is time, u_i are the components of the velocity vector, c is the speed of sound, and Δ is the total energy per unit mass. For an adiabatic flow $\chi = \gamma - 1$; for an isothermal flow $\chi = 0$, $\chi\Delta = 1$.

Plane flows with stationary streamlines are characterized by the differential relation

$$u_2 \frac{\partial u_1}{\partial t} - u_1 \frac{\partial u_2}{\partial t} = 0. \tag{1,4}$$

It is necessary to investigate the compatibility of the overdetermined system (1,1)–(1,4) (see ([1])). This class of flows includes stationary flows and nonstationary flows with plane or axial symmetry, where the stationary streamlines are rectilinear ([2,3]).

§ 2. Take as new unknown functions the velocity modulus v and θ, the angle of inclination of the velocity vector to the x_1-axis; then

$$u_1 = v\cos\theta, \qquad u_2 = v\sin\theta. \tag{2,1}$$

Substituting (2,1) into (1,4), we obtain

$$\frac{\partial\theta}{\partial t} = 0. \tag{2,2}$$

Let us write the equations of potential flows with stationary streamlines in curvilinear orthogonal coordinates q_1, q_2 ($q_2 =$ const are the streamlines, $q_1 =$ const are the lines orthogonal to them):

$$\frac{\partial\psi}{\partial t} + \frac{\partial\Delta}{\partial q_1} = 0, \qquad \frac{\partial\psi}{\partial q_2} = 0, \qquad \frac{\partial\Delta}{\partial q_2} = 0; \tag{2,3}$$

$$\frac{\partial\Delta}{\partial t} + 2\psi\frac{\partial\Delta}{\partial q_1}l - \psi^2\frac{\partial\psi}{\partial q_1}l^2 - \psi^3 m + c^2\left(l\frac{\partial\psi}{\partial q_1} + \psi n\right) = 0; \tag{2,4}$$

$$\frac{\partial}{\partial q_1}\left(\frac{1}{H_1}\frac{\partial H_2}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{1}{H_2}\frac{\partial H_1}{\partial q_2}\right) = 0, \tag{2,5}$$

where

$$\psi = vH_1, \qquad l = \frac{1}{H_1^2}, \qquad m = \frac{l}{2}\frac{\partial l}{\partial q_1}, \tag{2,6}$$

$$n = \frac{1}{2}\frac{\partial l}{\partial q_1} + \frac{l}{H_2}\frac{\partial H_2}{\partial q_1}, \qquad c^2 = \chi\left(\Delta - \frac{\psi^2}{2}\,l\right).$$

§ 3. Suppose that the Lamé coefficients are known and satisfy equation (2,5) (^{4}). Consider the compatibility of the system (2,3), (2,4). The func-

χ, Δ, and ψ depend only on q_1, t, while the functions l, m, n depend on q_1, q_2. The variable q_2 enters only into equation (2.4) as a parameter. Differentiating this equation with respect to q_2 and setting the differential consequences equal to zero, we obtain compatibility conditions for the functions $H_1(q_1, q_2)$, $H_2(q_1, q_2)$.

Theorem 1. *A plane unsteady potential flow with rectilinear stationary streamlines is a motion with plane or axial symmetry.*

Theorem 2. *A plane flow with curvilinear stationary streamlines, possessing a functional Jacobian, is either stationary or belongs to the class of solutions:*

1. $\chi = 0,\ \psi = 1,\ \Delta = \alpha t + \beta,\ \alpha, \beta$ are constants;
2. $\chi \neq 0,\ \psi = t^{-1},\ \Delta = q_1 t^{-2}$.

The functions H_1, H_2 are determined from a system of two differential equations (2.5) and, respectively, (3.1) or (3.2):

$$\alpha + n = m; \tag{3.1}$$

$$\chi n(q_1 - l/2) - m - 2(q_1 - l) = 0. \tag{3.2}$$

Theorem 3. *The functions* $\psi = t^{-1}$, $\Delta = q_1 t^{-2} + s$, *where for* $\chi \neq 2$, $s = s_0 t^{-4\chi/(\chi+2)}$; *for* $\chi = 2$, $s = -t^{-2}\ln t$; s_0 *is constant, describe a plane flow with curvilinear streamlines having a constant Jacobian. The Lamé coefficients* H_1, H_2 *are known functions of* q_1, q_2, *for example, for* $\chi = 2$

$$H_1^2 + H_2^2 = 1, \qquad H_1^{-2} + \ln(H_1^{-2} - 1) = 2(q_1 + q_2). \tag{3.3}$$

§ 4. From the condition of stationarity of the streamlines we have obtained an explicit dependence of the solutions on time in the variables q_1, q_2, t. This dependence is unchanged also in the variables x_1, x_2, t, so that the solution must have the form

$$u_i = \psi V_i, \qquad \Delta = -\psi' V + s, \tag{4.1}$$

where ψ, s are known functions of t; V and $V_i = \partial V/\partial x_i$ are functions only of x_1, x_2.

For $\chi = 0$, $\psi = 1$, $s = \alpha t + \beta$, for the function $V(x_1, x_2)$ from (1.2) we obtain the equation

$$(V_{iV}k - \delta_{ik}) \cdot V_{ik} = \alpha. \tag{4.2}$$

The function $V(x_1, x_2)$ for $\chi \neq 0$, $\psi = t^{-1}$, $s = 0$ must satisfy the equation

$$[V_{iV}k - \chi(V - V_i^2/2)\delta_{ik}]V_{ik} - 2V_i^2 + 2V = 0. \tag{4.3}$$

The Jacobian of the solution of equations (4.2) and (4.3) is two functions of one argument.

For $\chi = 2$, $\psi = t^{-1}$, $s = -t^{-2}\ln t$, integrating the system

$$\partial x_1/\partial q_1 = H_1 \cos\theta, \qquad \partial x_1/\partial q_2 = -H_2 \sin\theta,$$

$$\partial x_2/\partial q_1 = H_1 \sin\theta, \qquad \partial x_2/\partial q_2 = H_2 \cos\theta, \tag{4.4}$$

we obtain the solution

$$q_1 = x_1 + \frac{1}{2}(x_2^2 + 1) = V(x_1, x_2). \tag{4.5}$$

Similarly, for $\chi \neq 2,\ \psi = t^{-1},\ s = s_0 t^{-4\chi/(\chi+2)}$ we find

$$q_1 = \frac{1}{2}\left(x_2^2 + \frac{2-\chi}{2+\chi}x_1^2\right) = V(x_1, x_2). \tag{4.6}$$

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Note: Figure translations are in progress. See original paper for figures.