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G. G. Tsepina

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Abstract

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G. G. Tsepina

ON THE EMBEDDING PROBLEM FOR THE NILPOTENT PRODUCT OF FINITELY PRESENTED GROUPS

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In the works of K. A. Mikhailova (4,5) the embedding problem for the direct and free products of groups is solved. In particular, it is established that in the direct product of groups the embedding problem is undecidable (4), whereas in the free product it is decidable (5).

In the present paper the embedding problem is solved for the n-th nilpotent (3,6) product of groups, the numbering of the terms of the lower central series of a group being such that the direct product is the first nilpotent product.

We shall consider only finitely presented groups. It is known (8) that the n-th nilpotent product of finitely presented groups is a finitely presented group.

Theorem 1. *In the n-th nilpotent product of two free groups of rank two each (5), the embedding problem is undecidable.*

Theorem 2. *In the n-th nilpotent product of two groups, the weak embedding problem is decidable if in each of the given groups the weak embedding problem is decidable and every subgroup of either of them has a finite number of generators.*

For $n = 1$ the validity of the indicated theorems was proved in (4).

Let $\mathfrak{A} = \{a\}$ be the infinite cyclic group, $\mathfrak{B}$ the free group with positive alphabet $b_1, \dots, b_m$; denote by $\overline{\mathfrak{C}}$ their n-th nilpotent product. The group $\overline{\mathfrak{C}}$, as is known (3,6), can be represented in the form

$$\overline{\mathfrak{C}} = \mathfrak{A}(n)\overline{\mathfrak{B}} = \mathfrak{A}\overline{\mathfrak{B}}\overline{\mathfrak{U}},$$

where $\overline{\mathfrak{U}} = [\mathfrak{A}, \overline{\mathfrak{B}}]^{\overline{\mathfrak{C}}}$ is the normalized mutual commutant of the subgroups $\mathfrak{A}$ and $\overline{\mathfrak{B}}$ in the group $\overline{\mathfrak{C}}$, which is a torsion-free nilpotent group of class $[n/2]$ and has as generators those of the basic (9, p. 187) commutators of weights from one to n inclusive which are constructed on the ordered system of elements

$$a < b_1 < \dots < b_m,$$

whose weight is greater than one and in which, as a component, each contains the element a (^{8}). Denote the totality of such commutators by $g_1, \dots, g_p$, and in each of them replace every component a by the element a^α ($1 < \alpha$ is an integer); denote the commutators obtained in this way respectively by $Q_1, \dots, Q_p$. Then each of the commutators Q_i can be represented in the form

$$Q_i = g_i^{\xi_i} g_{i+1}^{\xi_{i+1}} \cdots g_p^{\xi_p}, \tag{1}$$

where ξ_i is some power of the number α, and $\xi_{i+1}, \dots, \xi_p$ are integral polynomials in α of degree not exceeding $n+1$, without constant terms, admitting effective determination, or some of them are equal to zero.

For the system of commutators $g_1, g_2, \dots, g_p$ we define the notion of a basic commutator as follows: 1) all basic commutators of weight $\cdots$

two shall be called basic; 2) suppose that all basic commutators of weights less than r have already been defined; then as basic commutators of weight r we shall call all commutators of the form $g_t = [g_l, a]$, where g_l is any basic commutator of weight $r-1$. The subgroup of the group $\mathfrak{C}$ generated by all basic commutators will be denoted by $\mathfrak{U}_0$. Now let the group $\mathfrak{B}$ be arbitrary with generators $b_1, \dots, b_m$, and let the group $\mathfrak{C}$ be the n-th nilpotent product of the groups $\mathfrak{A}$ and $\mathfrak{B}$, i.e.,

$$\mathfrak{C} = \mathfrak{A}(n)\mathfrak{B} = \mathfrak{A}\mathfrak{B}\mathfrak{U},$$

where $\mathfrak{U} = [\mathfrak{A}, \mathfrak{B}]^{\mathfrak{C}}$. The subgroup $\mathfrak{U}$, obviously, can be given by the same generators as $\mathfrak{U} \subset \mathfrak{C}$.

Lemma 1. *The subgroup $\mathfrak{B}\mathfrak{U}$ of the group $\mathfrak{C}$ is isomorphic to the factor group of the group $\mathfrak{R} = \mathfrak{B}(n-1)\mathfrak{U}_0$ by a normal divisor having, as a subgroup, a finite number of generators.*

Lemma 2. *If in the n-th nilpotent product of an infinite cyclic group and a group with solvable strong* $^{(5)}$ *membership problem (or membership problem) the strong membership problem is solvable, then the strong membership problem is also solvable in the n-th nilpotent product of any nilpotent group of class $k_1 \leq n$ and the given group.*

We omit the proofs of these lemmas.

Consider an arbitrary subgroup $\mathfrak{G}$ of the group $\mathfrak{C}$, generated by a finite number of elements which, as is known $^{(3)}$, can be uniquely represented in the form

$$a^{\alpha_i} U_i'', \quad a^{\beta_j} B_j' U_j', \tag{2}$$

where α_i, β_j are integers, $U_i'', U_j' \in \mathfrak{U}$; $B_j' \in \mathfrak{B}$. Let d be the greatest common divisor of the numbers α_i (if all $\alpha_i = 0$, put $d = 0$), and q the greatest common divisor of the numbers α_i and β_j (when all $\alpha_i = \beta_j = 0$, we shall take $q = 0$);

then the system (2) is equivalent to another system, which is obtained from (2) and has the form

$$a^d U_0^*, \quad a^q B_0 U_0, \quad B_1 U_1, \dots, B_k U_k, \tag{3}$$

where $U_0^*, U_0, \dots, U_k$ are words from $\mathfrak{U}$; $B_1, \dots, B_k$ are words from $\mathfrak{B}$.

Lemma 3. *If in the system of generators (3) of the subgroup* $\mathfrak{G}$ $d \neq 0$*, then the subgroup* $\mathfrak{G}^* = \mathfrak{G} \cap (\mathfrak{B}\mathfrak{U})$ *has a finite number of generators and there exists an algorithm for finding them.*

Theorem 3. *In the n-th nilpotent product of an infinite cyclic group and a group with solvable membership problem, the membership problem is solvable.*

We give a brief outline of the proof of this theorem. Suppose that in the group $\mathfrak{B}$ indicated above the membership problem is solvable, and prove the theorem by induction on the number n. For $n = 1$ the validity of this theorem was proved in (4). Now let $n > 1$, and suppose that for all $n_1 < n$ Theorem 3 has already been proved; we shall prove its validity for the group $\mathfrak{C} = \mathfrak{A}(n)\mathfrak{B}$.

From the induction hypothesis and Lemmas 1 and 2 it follows that the membership problem is solvable in the subgroup $\mathfrak{B}\mathfrak{U}$ of the group $\mathfrak{C}$. Consider an arbitrary element W' of the group $\mathfrak{C}$; then from the induction hypothesis it also follows that there exists an algorithm allowing one to determine either $W' \notin \mathfrak{G}$, or to find such a word $W \in \mathfrak{A}$ that $W' \in \mathfrak{G}$ if and only if $W \in \mathfrak{G}$, and moreover the canonical (8) expression of the element W contains only basic commutators of weight n among $g_1, \dots, g_p$.

For the system of generators of the subgroup $\mathfrak{G}$ two cases are possible: 1) in the system (3) $d = q = 0$ or $d \neq 0$; 2) in the system (3) $d = 0,\ q \neq 0$.

In the first case, the existence of the required algorithm follows from the solvability of the membership problem in the subgroup $\mathfrak{B}\mathfrak{U}$ and Lemma 3.

Suppose the second case holds; then, in order to construct the algorithm we need, we introduce an auxiliary subgroup $\mathfrak{G}^{(\alpha)}$, obtained from

by adjoining the element $a^{q\alpha}$ (α is a natural number). Then we construct two systems of "basic"commutators of weights from 1 to n, inclusive, on the following ordered sets of elements:

$$a^q < a^q B_0 U_0 < B_1 U_1 < \dots < B_k U_k,$$

$$a^{q\alpha} < a^q B_0 U_0 < B_1 U_1 < \dots < B_k U_k.$$

From the first (second) system we take all those commutators whose weight is greater than one and each of which contains, as a component, the element a^q ($a^{q\alpha}$). The set of all selected commutators of the first system we denote by $D_1, \dots, D_d$, and of the second by $L_{\alpha 1}, \dots, L_{\alpha d}$. It is not hard to verify that D_i

and $L_{\alpha i}$ are elements of $\mathfrak{U}$ and, as was noted above, $L_{\alpha i}$ can be written with the help of $D_i, \ldots, D_d$.

The following assertion holds: there exists an algorithm deciding the disjunction

$$(B_0 U_0)^\beta U \notin \mathfrak{G} \ \cup \ (B_0 U_0)^\beta U \in \mathfrak{G}, \tag{4}$$

where $\beta \neq 0$ is an integer and U is an arbitrary element of the subgroup $\mathfrak{U}$.

First step. As was noted above, there exists an algorithm that allows one to determine whether an element W belongs to the subgroup $\mathfrak{G}^{(1)}$. If $W \notin \mathfrak{G}^{(1)}$, then $W \notin \mathfrak{G}$. If, however, $W \in \mathfrak{G}^{(1)}$, then by means of a finite number of transformations it can be represented in the form

$$W = W_1'(B_0 U_0)^{\xi_1} W_1,$$

where ξ_1 is an integer, $W_1' \in \mathfrak{G}$, and W_1 is written in "canonical" form with the help of $D_1, \ldots, D_d$. If $\xi_1 \neq 0$, the process terminates by applying relation (4).

Second step. Let $\xi_1 = 0$ and $W_1 = D_{l_1}^{\rho_{11}} \cdots D_d^{\rho_{1d}}$; then, considering the subgroup $\mathfrak{G}^{(\alpha_1)}$, where $\rho_{1 l_1} \not\equiv 0 \pmod{\alpha_1}$, we obtain

$$W_1 \notin \mathfrak{G}^{(\alpha_1)} \ \cup \ W_1 \in \mathfrak{G}^{(\alpha_1)}.$$

Consider only the case $W_1 \in \mathfrak{G}^{(\alpha_1)}$; then W_1 can be represented in the form

$$W_1 = W_2'(B_0 U_0)^{\xi_2} W_2, \tag{5}$$

where $W_2' \in \mathfrak{G}$, and W_2 is written in "canonical" form with the help of the commutators $L_{\alpha_1 1}, \ldots, L_{\alpha_1 d}$. The case $\xi_2 \neq 0$ was considered above; therefore we shall assume that $\xi_2 = 0$ and $W_2 \neq E$. Using relation (4) from (*), theorem 5 from (**) and the relation (1) indicated above, W_2 can be represented in the form

$$W_2 = D_{r_1}^{\delta_{2r_1}} \cdots D_d^{\delta_{2d}}.$$

From relation (5) we have

$$R_1 = W_1 W_2^{-1} = D_{p_1}^{\sigma_{1p_1}} \cdots D_d^{\sigma_{1d}} \in \mathfrak{G},$$

where $\sigma_{1p_1} \neq 0$ and $p_1 = \min(r_1, l_1)$. For further consideration we take $\overline{W}_2 = W_2$, if $r_1 > l_1$, and $\overline{W}_2 = W_1$ otherwise.

Suppose that at the s-th step the process has not terminated; then we have words

$$R_i = D_{l_i}^{\sigma_{ip_i}} \cdots D_d^{\sigma_{id}} \in \mathfrak{G} \quad (i = 1, \dots, f;\ p_i \neq p_j \text{ for } i \neq j)$$

and

$$\overline{W}_s = D_{l_s}^{\rho_{sl_s}} \cdots D_d^{\rho_{sd}} \quad (l_s \geq l_1),$$

where

$$W \in \mathfrak{G} \Longleftrightarrow W_s \in \mathfrak{G}.$$

At the $(s+1)$-st step, choosing α_s in the corresponding way, we shall have

$$\overline{W}_s \notin \mathfrak{G}^{(\alpha_s)} \ \cup\ \overline{W}_s \in \mathfrak{G}^{(d_s)}.$$

If $\overline{W}_s \in \mathfrak{G}^{(\alpha_s)}$, then, as above,

$$\overline{W}_s = W'_{s+1}(B_0 U_0)^{\zeta_{s+1}} W_{s+1},$$

where $W'_{s+1} \in \mathfrak{G}$,

$$W_{s+1} = D_{r_s}^{\delta(s+1)r_s} \dots D_d^{\delta(s+1)d}.$$

The process terminates at this stage in the case when $\zeta_{s+1} \neq 0$, or else W_{s+1} is expressed by means of the words R_i. In the contrary case one can again find words

$$\overline{W}_{s+1} = D_{l_{s+1}}^{\rho(s+1)l_{s+1}} \dots D_d^{\rho(s+1)d}$$

$(l_{s+1} \geq l_s)$ and, for $\min(l_s, r_s) \neq p_i$,

$$R_{f+1} = D_{p_{f+1}}^{\sigma(f+1)p_{f+1}} \dots D_d^{\sigma(f+1)d},$$

such that $R_{f+1} \in \mathfrak{G}$ and $W \in \mathfrak{G} \Longleftrightarrow \overline{W}_{s+1} \in \mathfrak{G}$.

Such a process is, obviously, finite.

From Theorem 3 and Lemma 2 there follows immediately

Theorem 4. *In the n-th nilpotent product of a nilpotent group and a group with solvable membership problem, the membership problem is solvable.*

Lemma 4. *The polydirect product* (9) *of two groups with a finite number of generators and having finitely separable* (1) *subgroups is a group with finitely separable subgroups.*

Theorem 5. *If the groups $\mathfrak{M}$ and $\mathfrak{N}$ have finitely separable subgroups, then their n-th nilpotent product is a group with finitely separable subgroups.*

In a finitely presented group with finitely separable subgroups, as is known (1), the membership problem is solvable; therefore

Corollary. *In the n-th nilpotent product of two groups with finitely separable subgroups, the membership problem is solvable.*

Moscow State Pedagogical
Institute named after V. I. Lenin

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Note: Figure translations are in progress. See original paper for figures.

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