ON $C\Phi$-KERNELS OF SETS EFFECTIVELY DIFFERENT FROM Φ-SETS

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Abstract

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ON $C\Phi$-KERNELS OF SETS EFFECTIVELY DIFFERENT FROM Φ-SETS

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As in ($^{3-5}$), here we study certain questions of descriptive set theory by means of P. S. Novikov' s idea of effective difference ($^{1-3}$). In (^{4,5}) it was proved that if a set T of a space E is effectively different from all Φ-sets of this space, then both T and its complement $E - T = CT$ contain discontinua, and also contain absolute C_σ-sets not separable, respectively, from $E - T$ and T by any absolute F_σ-sets. Here it will be proved that T contains kernels of considerably more complicated structure–the so-called $C\Phi$-kernels, i.e., absolute $C\Phi$-sets not separable from $E - T$ by any absolute Φ-sets.

1°. To the symbols $\Pi(E)$, $\Pi_\Phi(E)$, $F(E)$, $Pt(E)$ we assign the same meaning as in (^{3}), pp. 129-130. The set $\Pi_\Phi(E)$ is called the Φ-base of the space E. The definition itself of a set effectively different from Φ-sets is as follows:

Definition. Let E be a metric space, Φ an arbitrary δs-operation. We shall say that a set T, $T \subseteq E$, is **effectively different from all Φ-sets of the space** E, if there exist a compactum Z, $0 \subset Z \subseteq E$, and a mapping ν of some Φ-base $\Pi_\Phi(E)$ into the set Z, for which the following conditions are fulfilled: a) for every sequence of closed sets $\{F_n\} \in \Pi_\Phi(E)$ we have

$$\nu\{F_n\} \in T \cdot C\Phi\{F_n\} + CT \cdot \Phi\{F_n\};$$

b) $Pt(Z) \subseteq \Pi_\Phi(E)$; c) the mapping ν is continuous on the metric space $\Pi t(Z)$.

In this case we call ν a **distinguishing function** for T, and the compactum Z the **metric base** of the function ν.

Comparing this definition with the basic definition in (^{3}), p. 135, we note, first, that they differ only in that in (^{3}) it is assumed with respect to Z that it is a bounded closed subset of the space E, whereas here we require that Z be a compactum. Secondly, for a Euclidean space E the two definitions coincide completely. Thirdly, both the existence theorem for a set T effectively different from all Φ-sets of the space E ((^{3}), $p.$146) and all properties of this set T found in ($^{3-6}$) are completely preserved if one assumes that Z is a compactum. In this case only the metric of the space Z is used, and not at all that of the space E, so that everywhere one may regard E as a topological space containing a metric compactum as a subspace.

2°. In what follows, by Φ we shall mean a δs-operation with the following property:

(τ). All chains of the operation Φ are infinite, and for every n_0 there exists a chain $\{n_1, n_2, ...\}$ of the operation Φ such that

$$n_0 < n_1 < n_2 < \cdots.$$

For an operation Φ with this property, evidently,

$$\Phi\{F_1, \dots, F_{n_0}; M, \bar{M}, M, \dots, M, ...\} = M.$$

All operations that are stronger than the operation of the lower limit, in particular the operations $B_\alpha (\alpha \geq 2)$ and A, possess property (τ).

Theorem 1. If a set T, situated in the space E, is effectively distinct from all Φ-sets of this space, then there exists a perfect compact set Z_1, $0 \subset Z_1 \subset E$, such that the intersection $T \cdot Z_1$ is a $C\Phi$-set of the space Z_1, effectively distinct from all Φ-sets of this space.

Proof. Let ν be a distinguishing function for T, and let Z be a metric basis of the function ν. From the compactness of Z it follows that $\Pi t(Z)$ is also compact. Consequently, the image of the space $\Pi t(Z)$ under the continuous mapping ν, which we denote by Z_1, is also compact. According to Definition II.1°, $Z_1 \subset Z$.

The sets $Z_1 \cdot T$ and $Z_1 \cdot (E - T) = Z_1 \cdot CT$ are nonempty. Indeed, $\nu\{Z, Z, \dots, Z, ...\} \in Z_1 \cdot (T \cdot C\Phi\{Z\} + CT \cdot \Phi\{Z\}) = Z_1 \cdot CT$, since $Z_1 \cdot CT \neq 0$. On the other hand, $\nu\{0, 0, \dots, 0, ...\} \in Z \cdot (T \cdot C\Phi\{0\} + CT \cdot \Phi\{0\}) = Z \cdot T$. Consequently, also $Z \cdot T \neq 0$. If now by F^0 we denote any nonempty closed set contained in $Z \cdot T$, then

$$\nu\{F^0, F^0, \dots, F^0, ...\} \in Z_1 \cdot (T \cdot C\Phi\{F^0\} + CT \cdot \Phi\{F^0\}) =$$

$$= Z_1 \cdot T \cdot C\Phi\{F^0\}, \qquad (*)$$

which proves the nonemptiness of the set $Z_1 \cdot T$.

Next, let $z = \nu\{F_1, F_2, \dots, F_n, ...\}$ be an arbitrary point of the set Z_1 ($\{F_n\} \in \Pi t(Z)$), and let $U(z)$ be any neighborhood (relative to the space E) of this point. From the continuity of the function ν at the point $\{F_n\}$ it follows that, for some n_0, $\nu\{F_1, \dots, F_n; Q_1, Q_2, \dots, Q_n, ...\} \in U(z)$, whatever the point $\{Q_n\} \in \Pi t(Z)$ may be. At the same time, for the closed set F^0, $0 \subset F^0 \subset TZ$, $\nu\{F_1, \dots, F_n; F^0, F^0, \dots, F^0, ...\} \in Z_1 \cdot T \cdot C\Phi\{F^0\}$ (see ($*$), and also property (τ) of the operation Φ), while $\nu\{F_1, \dots, F_n; Z, Z, \dots, Z, ...\} \in Z_1 \cdot CT$. Thus, in an arbitrary neighborhood $U(z)$ of any point $z \in Z_1$ there are both points of $T \cdot Z_1$ and points of $CT \cdot Z_1$, and the compact set Z_1 is perfect.

The set $T \cdot Z_1$ is effectively distinct from all Φ-sets of the space Z_1. We construct a distinguishing function ν_1 for $T \cdot Z_1$ as follows. As its domain we take the Φ-basis $\Pi_\Phi(Z_1)$, which is obtained by adjoining to the set $\Pi t(Z_1)$ the single sequence $\{0, 0, \dots, 0, \dots\}$ (see (3), p. 136, Remark 1), and as metric basis the compact set Z_1 itself. Denoting by z_0 some fixed point of the set $Z_1 \cdot T$, we put

$$\nu_1\{F_n\} = \begin{cases} \nu\{F_n\}, & \text{for } \{F_n\} \in \Pi t(Z_1), \\ z_0, & \text{for } \{F_n\} = \{0, 0, \dots, 0, \dots\}. \end{cases}$$

It is easy to see that ν_1 is indeed a distinguishing function for $T \cdot Z_1$ (relative to the space Z_1).

It remains to prove that $CT \cdot Z_1$ is a Φ-set of the space Z_1. We turn to the continuous mapping ν of the space $\Pi t(Z)$ onto Z_1 and define diagonal sets L_n, $L_n \subset Z_1$, for each natural number n in the following way. A point $x \in Z_1$ belongs to L_n if and only if among all sequences $\{F_1^x, F_2^x, \dots, F_n^x, \dots\} \in \Pi t(Z)$ that are preimages of the point x under the mapping ν, there is at least one such sequence for which $x \in F_n^x$.

The sets $L_n \neq 0$, for, obviously, $\nu\{Z, Z, \dots, Z, \dots\} \in \prod_{n=1}^{\infty} L_n$.

Let $x^0 = \lim_{k\to\infty} x_k$, where $x_k \in L_n$ for all k (n fixed).

From the definition of L_n, by virtue of the compactness of $\Pi t(Z)$, there follows the existence of a sequence $x_k^1, x_k^2, \dots, x_k^p, \dots$, for which:

1) $\{x_k^1, x_k^2, \dots, x_k^p, \dots\}$ is a subsequence of the sequence $\{x_1, \dots, x_k, \dots\}$;
2) in the space $\mathrm{Pt}(Z)$ one can select a convergent sequence of its points $\{F_m^1\}, \{F_m^2\}, \dots, \{F_m^p\}, \dots$, such that $\nu\{F_m^p\} = x_k^p$ and $x_k^p \in F_n^p$ ($p = 1, 2, \dots$). Denoting $\lim_{p\to}\{F_m^p\} = \{F_m^0\}$, by virtue of the continuity of ν at the point $\{F_m^0\}$ we have $\nu\{F_m^0\} = x^0$. Hence, in the space $F(Z)$,

$$\lim_{p\to\infty} F_m^p = F_m^0 \quad (m = 1, 2, \dots).$$

In particular, for the n fixed above, $\lim_{p\to\infty} F_n^p = F_n^0$. This, as well as the fact that $x_k^p \in F_n^p$ for every p and that $\lim_p x_k^p = x^0$, leads to the conclusion $x^0 \in F_n^0$, consequently, $x^0 \in L_n$. Thus L_n is closed for every n.

Let us show that $\Phi\{L_n\} = CT \cdot Z_1$. Take an arbitrary point $x \in Z_1$. We denote every point of its complete preimage $\nu^{-1}(x)$ under the mapping ν of $\mathrm{Pt}(Z)$ onto Z_1 by $\{F_1^x, F_2^x, \dots, F_n^x, \dots\}$. By the symbol $\{F_1(x), F_2(x), \dots, F_n(x), \dots\}$ we denote any of those points $\{F_1^x, F_2^x, \dots, F_n^x, \dots\}$ which are constructed in the following way: if, for a given n (and for the fixed x above), for every F_n^x we have $x \notin F_n^x$, then as $F_n(x)$ we choose any F_n^x; if, however, there exist F_n^x for which $x \in F_n^x$, then any of precisely these F_n^x is denoted by $F_n(x)$. Thus such a part of the set $\nu^{-1}(x)$, all points $\{F_1(x), F_2(x), \dots, F_n(x), \dots\}$ of which possess the following property, has been singled out: for every n, either $x \in F_n(x)$ for all $F_n(x)$, or for every F_n^x one has $x \notin F_n^x$. From this property of the $\{F_n(x)\}$ and from the definition of the diagonal sets L_n it follows that

$$x \in L_n \cdot F_n(x) + (E - L_n) \cdot (E - F_n(x)).$$

Hence, by virtue of the definition of the δs-operation Φ,

$$x \in \Phi\{L_n\} \cdot \Phi\{F_n(x)\} + C\Phi\{L_n\} \cdot C\Phi\{F_n(x)\}. \tag{1}$$

On the other hand,

$$x = \nu\{F_n(x)\} \in Z_1 \cdot CT \cdot \Phi\{F_n(x)\} + Z_1 \cdot T \cdot C\Phi\{F_n(x)\}. \tag{2}$$

From comparing assertions (1) and (2), valid for any point $x \in Z_1$, we easily infer that $Z_1 \cdot CT = \Phi\{L_n\}$, $Z_1 \cdot T = Z_1 - \Phi\{L_n\}$. The theorem is proved.

Remark 1. If in the statement of the theorem just proved the word "perfect" is omitted, then it will already be true for any δs-operation Φ, and not only for an operation with property (τ).

Remark 2. All discontinuums constructed by us in ([3]–[5]) (see the beginning of the present article) are subsets of the compactum Z_1, which is the image of $\mathrm{Pt}(Z)$ under the mapping ν.

3°. In ([6]), p. 298, it is proved that if the δs-operation Φ is stronger than the δs-operation Ψ and the set T, $T \subset E$, is effectively distinct from all Φ-sets of the space E, then T is also effectively distinct from all Ψ-sets of this space. In view of this, from Theorem 1 for δs-operations Φ and Ψ with property (τ) there follows the following

Theorem 2. *If the set T, $T \subset E$, is effectively distinct from all Φ-sets of the space E and the operation Φ is stronger than the operation Ψ, then there exists a perfect compactum Z_1^Ψ, $0 \subset Z_1^\Psi \subseteq E$, such that the intersection $T \cdot Z_1^\Psi$ is a $C\Psi$-set of the space Z_1^Ψ, effectively distinct from all Ψ-sets of this space.*

Let us note that, as is seen from the proof of Theorem 1 and from the proof mentioned in ([6]), p. 298, the compacta Z_1 and Z_1^Ψ in Theorems 1 and 2 respectively can be chosen so that $Z_1^\Psi \subseteq Z_1$.

We now apply Theorem 2 to sets effectively distinct from A-sets. Using the known properties of Borel sets, as well as the fact that the A-operation is stronger than the operation B_α, which yields B-sets

of class α, we obtain, as a consequence of Theorem 2, the following proposition.

Theorem 3. *If a set (in particular, a CA-set) T, $T \subset E$, is effectively distinct from all A-sets of the space E, then for every ordinal α, $2 \leq \alpha < \Omega$, there exists a perfect compactum Z_1^α, $0 \subset Z_1^\alpha \subseteq E$, such that the intersection $Z_1^\alpha \cdot T$ is a CB_α-set (and consequently a $B_{\alpha+1}$-set) of the space Z_1^α, effectively distinct from all B_α-sets of this space Z_1^α.*

Remark to 2^0 remains valid also for Theorems 2 and 3. Omitting the word "perfect" from the formulation of the theorem, we may assume that it is also true for $\alpha = 0; 1$.

From Theorems 1 and 3 there follows the following proposition.

Corollary. *Under the hypotheses of Theorem 3, both the set T and its complement $E-T$, for every $\alpha < \Omega$, contain absolute B_α-sets not separable respectively from $E-T$ and T by any absolute B_β-sets, where $\beta < \alpha$. Moreover, T contains an absolute CA-set not separable from $E-T$ by any absolute A-set, while $E-T$ contains an absolute A-set not separable from T by any absolute B-set.*

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.