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Abstract

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ON A MODIFICATION OF THE CONCEPT OF THE MODULUS OF SMOOTHNESS AND ITS APPLICATION TO ESTIMATING FOURIER COEFFICIENTS

(Presented by Academician S. N. Bernstein on 15 VII 1964)

Let $f(x) \in L_{2\pi}$. Put

$$ {}^{s}\Delta_t^p f(x) = \sum_{k=0}^{p} (-1)^k C_p^k f[x + (p-2k)t]. $$

Definition. Put*

$$ L^{(p)}(h, x, f) = \frac{1}{h} \int_0^h {}^{s}\Delta_t^p f(x)\, dt. $$

The quantity

$$ L^{(p)}(h, f) = \sup_{-\pi \le x \le \pi} \left| L^{(p)}(h, x, f) \right| $$

will be called the L-modulus of smoothness of order p of the function f.

Theorem 1. Let $f(x) \in \mathcal{L}_{2\pi}^2$,

$$ f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx). $$

Then

$$ \frac{1}{n^{2p}} \sum_{k=1}^{n} (a_k^2 + b_k^2) k^{2p} \le C(p) \int_{-\pi}^{\pi} \left[L^{(p)}\left(\frac{1}{n}, x, f\right) \right]^2 dx, $$

where p is an arbitrary natural number, and $C(p)$ depends only on p.

Proof. It is easy to see that for even p

$$ {}^{s}\Delta_t^p f(x) \sim (-1)^{p/2} 2^p \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \sin^p kt, $$

$$ n \int_0^{1/n} {}^{s}\Delta_t^p f(x)\, dt \sim (-1)^{p/2} 2^p \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\, n \int_0^{1/n} \sin^p kt\, dt, $$

and for odd p

$$ n \int_0^{1/n} {}^{s}\Delta_t^p f(x)\, dt \sim (-1)^{(p-1)/2} 2^p \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx)\, n \int_0^{1/n} \sin^p kt\, dt. $$

By Parseval' s equality,

$$ \frac{1}{\pi} \int_{-\pi}^{\pi} \left[L^{(p)} \left(\frac{1}{n}, x, f \right) \right]^2 dx = C_1(p) \sum_{k=1}^{\infty} (a_k^2 + b_k^2) \left(n \int_0^{1/n} \sin kt\, dt \right)^2, $$

whence

$$ \frac{1}{n^{2p}} \sum_{k=1}^{n} k^{2p} (a_k^2 + b_k^2) \leq C(p) \int_{-\pi}^{\pi} \left[L^{(p)} \left(\frac{1}{n}, x, f \right) \right]^2 dx. $$

The theorem is proved.

Corollary 1. If $f(x) \in \mathcal{L}_{2\pi}^2$, then

$$ |a_n| = O \left(\left\{ \int_{-\pi}^{\pi} \left[L^{(p)} \left(\frac{1}{n}, x, f \right) \right]^2 dx \right\}^{1/2} \right). $$

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* Relying on well-known results (([1]), pp. 83-84), it is not difficult to show that $L^{(p)}(h, x, f)$ is a function summable in x on $[0, 2\pi]$ for $f(x) \in \mathcal{L}_{2\pi}$ and belongs to $\mathcal{L}_{2\pi}^2$ for $f(x) \in \mathcal{L}_{2\pi}^2$.

In particular,

$$ |a_n| = O \left(\left\{ L^{(1)} \left(\frac{1}{n}, f \right) \right\} \right). \tag{1} $$

From this we obtain the well-known estimate

$$|a_n| = O\left(\omega\left(\frac{1}{n}, f\right)\right). \tag{2}$$

The same estimates are also valid for b_n.

Example.

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos 2^k x}{k^{1+\alpha}}, \qquad 0 < \alpha < 1,$$

which belongs to A. Zygmund (([1]), pp. 294—296), shows that

$$L^{(1)}\left(\frac{1}{n}, f\right) = O\left(\frac{1}{|\ln n|^{1+\alpha}}\right) = O\left(\frac{1}{\ln n}\right),$$

whereas

$$\omega\left(\frac{1}{n}, f\right) \neq O\left(\frac{1}{\ln n}\right).$$

Thus inequality (1) is sharper than (2).

Corollary 2. If

$$\int_{-\pi}^{\pi} \left[L^{(p)}\left(\frac{1}{n}, x, f\right)\right]^2 dx = O\left(\frac{1}{n^{2p}}\right),$$

then

$$a_k = O\left(\frac{1}{k^p}\right), \qquad b_k = O\left(\frac{1}{k^p}\right).$$

Corollary 3. If $|a_{k+1}| \le |a_k|$, then

$$|a_n| = O\left(\frac{\left\{\int_{-\pi}^{\pi} \left[L^{(p)}\left(\frac{1}{n}, x, f\right)\right]^2 dx\right\}^{1/2}}{\sqrt{n}}\right).$$

Analogous assertions are also valid for b_n.

Theorem 2. In order that a function $f(x)$ whose Fourier series is lacunary have a derivative of order p belonging to $\operatorname{Lip}\alpha$ $(0<\alpha<1)$ or to Z $(\alpha=1)$, it is necessary and sufficient that its Fourier coefficients have order $1/n^{p+\alpha}$.

Proof. It is well known that the necessity of the condition holds even when the Fourier series is not lacunary. For a lacunary series satisfying the conditions of the theorem,

$$\sum_{k=n}^{\infty}(|a_k|+|b_k|)=O\left(\frac{1}{n^{p+\alpha}}\right).$$

All the more,

$$E_n=O\left(\frac{1}{n^{p+\alpha}}\right).$$

Hence, taking into account the theorems of S. N. Bernstein and A. Zygmund ([2], pp. 138–139, 145), we obtain the sufficiency of the condition.

Remark. For $p=0$ $(0<\alpha<1)$ this theorem is known ([1], p. 691).

Corollary. It is known that if $f(x)$ has bounded variation, then

$$|a_k|=O\left(\frac{1}{k}\right),\qquad |b_k|=O\left(\frac{1}{k}\right).$$

It is shown by an example ([1], p. 203) that the order of this estimate cannot be improved, even if $f(x)$ is assumed continuous. Moreover, we shall show that the indicated order cannot be improved even if the function of bounded variation $f(x)\in Z$.

Example. Let

$$f(x)=\sum_{m=1}^{\infty}\frac{1}{2^m}\cos 2^m x.$$

Lorentz ([3]) proved that if

$$\sum_{k=n}^{\infty}(|a_k|+|b_k|)=O\left(\frac{1}{n}\right),$$

then $f(x)$ has bounded variation. Applying Lorentz' s theorem and Theorem 2, it is easy to see that $f(x)$ has bounded variation and belongs to Z, whereas its coefficients are not $o(1/k)$.

Theorem 3. Let $f(x)\in L^2_{2\pi}$ and

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + d_k \sin kx),$$

where* $a_k \geqslant 0$. Then, for even p,

$$\frac{1}{n^p} \sum_{k=1}^{n} a_k k^p \leqslant C(p) \left| L^{(p)} \left(\frac{1}{n}, 0, f \right) \right|,$$

where $C(p)$ is a constant depending only on p.

Proof. For even p we have

$${}^{s}\Delta_t^p f(0) = (-1)^{p/2} 2^p \sum_{k=1}^{\infty} a_k \sin^p kt,$$

$$\left| n \int_0^{1/n} {}^{s}\Delta_t^p f(0)\, dt \right| \geqslant 2^p \sum_{k=1}^{n} a_k n \int_0^{1/n} \sin^p kt\, dt,$$

$$\frac{1}{n^p} \sum_{k=1}^{n} a_k k^p \leqslant C(p) \left| L^{(p)} \left(\frac{1}{n}, 0, f \right) \right|.$$

Corollary. If $a_k \downarrow 0$, then

$$a_n = O \left(\frac{L^{(p)} \left(\frac{1}{n}, 0, f \right)}{n} \right).$$

In particular, if $f(x)$ has m derivatives, and $f^m(x)$ belongs to $\mathrm{Lip}\, \alpha$ $(0 < \alpha < 1)$ or Z $(\alpha = 1)$, then

$$a_n = O \left(\frac{1}{n^{1+m+\alpha}} \right).$$

For functions of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx$$

with $m = 0,\ 0 < \alpha < 1$, this fact was established by Lorentz ([3]).

Thus, if $f(x)$ has m derivatives and $f^{(m)}(x) \in \mathrm{Lip}\, 1$, then $a_n = O(1/n^{2+m})$.

The example of the function

$$\Psi(x) = \sum_{k=1}^{\infty} \frac{\cos kx}{k^2}$$

shows that this estimate cannot be improved even for $m = 0$.

* No restrictions are imposed on b_k.

Theorem 4. Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx),$$

where $b_k \geqslant 0$. Then, for odd p, we have

$$\frac{1}{n^p} \sum_{k=1}^{n} k^p b_k \leqslant C(p) \left| n^2 \int_0^{1/n} \left[\int_0^u {}^s\Delta_t^p f(0)\, dt \right] du \right|.$$

Proof. For odd p,

$${}^s\Delta_t^p f(0) = (-1)^{(p-1)/2} 2^p \sum_{k=1}^{\infty} b_k \sin^p kt,$$

$$\int_0^u {}^s\Delta_t^p f(0)\, dt = (-1)^{(p-1)/2} 2^p \sum_{k=1}^{\infty} b_k \int_0^u \sin^p kt\, dt,$$

$$\left| \int_0^{1/n} \left[\int_0^u {}^s\Delta_t^p f(0)\, dt \right] du \right| \geqslant C_1(p) \sum_{k=1}^{n} b_k \int_0^{1/n} \left[\int_0^u \sin^p kt\, dt \right] du,$$

$$\frac{1}{n^p} \sum_{k=1}^{n} k^p b_k \leqslant C(p) \left| n^2 \int_0^{1/n} \left[\int_0^u {}^s\Delta_t^p f(0)\, dt \right] du \right|.$$

The theorem is proved.

Corollary. If $b_k \downarrow 0$, then

$$b_n = O\left(\left| n \int_0^{1/n} \left[\int_0^u {}^s\Delta_t^p f(0)\, dt \right] du \right| \right).$$

In particular, if $f(x)$ has m derivatives, and $f^{(m)}(x) \in \operatorname{Lip} \alpha$ $(0 < \alpha < 1)$ or Z $(\alpha = 1)$, then

$$b_k = O\left(\frac{1}{k^{1+m+\alpha}}\right).$$

For

$$f(x) = \sum_{k=0}^{\infty} b_k \sin kx$$

with $m = 0,\ 0 = m < \alpha < 1$, this fact was established by Lorentz ([3]).

Theorem 5. Let

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx),$$

where $a_k \downarrow 0,\ b_k \downarrow 0$. In order that $f(x)$ have a derivative of order p belonging to $\mathrm{Lip}\,\alpha$ $(0 < \alpha < 1)$ or Z $(\alpha = 1)$, it is necessary and sufficient that

$$a_k = O\left(\frac{1}{k^{1+p+\alpha}}\right), \qquad b_k = O\left(\frac{1}{k^{1+p+\alpha}}\right).$$

The necessity of the condition is clear from the corollaries to Theorems 3 and 4. The sufficiency is proved analogously to the sufficiency in Theorem 2.

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CITED LITERATURE

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[3] G. G. Lorentz, *Math. Zs.*, **51**, 135 (1948).

Note: Figure translations are in progress. See original paper for figures.