ON QUASIANALYTIC FUNCTIONS IN \(L(-\infty,\infty)\)

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Abstract

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MATHEMATICS

S. P. GEISBERG

ON QUASIANALYTIC FUNCTIONS IN $L(-\infty, \infty)$

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1. Usually, in order to obtain necessary and sufficient conditions for the quasianalyticity of a function, one has to consider quasianalytic classes of functions ([1,2]); this is caused by the difficulties that arise in proving that the corresponding sufficient conditions are also necessary. Thus, for example, according to Mandelbrojt ([2]), the class $L(M_n)$ of functions $f(x)$ infinitely differentiable in the space* and defined by the conditions $\|f^{(n)}\|_L \leq M_n$, is called quasianalytic if, for every function $f \in L(M_n)$, the closures of the linear spans of its derivatives $f^{(n)}$ and of its translates $f_x(t) = f(x+t)$ coincide. The question naturally arises of such a definition of the concept of a quasianalytic function which would make it possible to give necessary and sufficient conditions of quasianalyticity referring to a function, and not to a class of functions. In this note it is shown that for L one can propose the following definition.

Definition. A function $f \in L$ is called **A-quasianalytic** if, for every functional $m \in M$ and every $\{x_k\}$, $x_k \neq x_0$, $x_k \to x_0$, from the conditions

$$m(f_{x_k}) = \int_{-\infty}^{\infty} f(x_k + t) m(t)\, dt = 0, \qquad k = 1, 2, \dots, \infty,$$

it follows that $m(f_x) = 0$.

Obviously, every function belonging to a quasianalytic class of Mandelbrojt is A-quasianalytic. The properties of A-quasianalytic functions are established in the following theorems.

Theorem 1. *If f is an A-quasianalytic function, then f is infinitely differentiable, $f^{(n)} \in L(-\infty, \infty)$, and** *

$$\lim_{\tau \to 0} \left\| \left(f_\tau - \sum_{j=0}^{n} \frac{\tau^j}{j!} f^{(j)} \right) \tau^{-(n+1)} - \frac{f^{(n+1)}}{(n+1)!} \right\|_L = 0,$$

$$f^{(0)} = f, \qquad n = 0, 1, \dots, \infty. \tag{1}$$

Theorem 2. *Let $f(t)$ be an even function, $f \in L$, and let its Fourier transform $F(x)$ satisfy the conditions:*

1°. *$F(x) > 0$, is differentiable, and for $x > 0$, $-xF'(x)/F(x)$ strictly increases.*

2°. *For some $\beta > 0$, $|F'(x)| = O(|x|^\beta + 1)$.*

Then, in order that $f(t)$ be an A-quasianalytic function, it is necessary and sufficient that the following conditions hold:

3°. *$f(t)$ is infinitely differentiable, $f^{(n)} \in L$, $n = 1, 2, \dots, \infty$, and (1) holds.*

4°.

$$\int_1^\infty \frac{\log F(x)}{x^2} = -\infty.$$

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* In what follows, L and M denote the real space $L(-\infty, \infty)$ and its conjugate.

** Formula (1) should be regarded as a recurrence relation defining the successive derivatives of the function f.

Conditions $1^\circ, 2^\circ$ occur in ([2]). Theorem 2, together with ([2]), p. 52, Lemma 1; p. 38, Theorem 1, shows that for functions satisfying conditions $1^\circ, 2^\circ$, A-quasianalyticity and quasianalyticity in the sense of Mandelbrojt coincide. Theorems 1 and 2 may also be regarded as a development of Wiener' s theorem ([3]), since they give conditions for closedness in L of any system of translates f_x having a limit point $x_k \to x_0$.

2. Proof of Theorem 1. We restrict ourselves to the case $n = 1$. For $\tau > 0$ put $\varphi_\tau = (f_\tau - f)/\tau$, and first show that $\|\varphi_\tau\|_L = O(1)$. Indeed, otherwise it is not difficult to show that $\psi_\tau = \varphi_\tau / \|\varphi_\tau\|_L$ tends weakly to zero as $\tau \to 0+$. Then, by induction, one can construct a sequence of functionals $m_{\tau_k} \in M$ such that

$$m_{\tau_0}(\psi_{\tau_0}) = 1; \quad m_{\tau_k}(f) = 0, \ k = 0, 1, \dots, \infty; \quad m_{\tau_k}(\psi_{\tau_j}) = 0, \quad 0 \le j < k;$$

$$m_{\tau_k}(\psi_{\tau_k}) = -\sum_{j=0}^{k-1} m_{\tau_j}(\psi_{\tau_k}); \quad \|m_{\tau_k}\|_M < 2^{-k},$$

where $\{\psi_{\tau_k}\}$ is a suitably chosen subsequence of $\{\psi_\tau\}$, $\tau_k \downarrow 0$. Consequently, for the functional $m = \sum m_j$ we have $m \in M$ and $m(f_{\tau_0}) = m(\psi_{\tau_0}) = 1$, $m(f_{\tau_k}) = m(\psi_{\tau_k}) = 0$, $k = 1, 2, \dots, \infty$, which contradicts the A-quasianalyticity of the function f.

Next, without loss of generality, one may suppose that for some σ, $F(x) \neq 0$ for $|x| \le \sigma$. Denote by $\sigma_{\alpha,\beta}(z)$ the entire function of degree σ for which $\sigma_{\alpha,\beta}(0) = \alpha$,

$\sigma'_{\alpha,\beta}(0)=\beta$, and $\|\sigma_{\alpha,\beta}(x)\|_L<\infty$. From the Wiener-Paley theorems ([4]) and Wiener ([3]) it follows that there exists $m_{\alpha,\beta}\in M$ such that $\sigma_{\alpha,\beta}(x)\equiv m_{\alpha,\beta}(f_x)$. We show that this implies the weak convergence of $\{\varphi_\tau\}$ as $\tau\downarrow 0$. In fact, since $\|\varphi_\tau\|_L=o(1)$, otherwise there would exist a functional $m\in M$, subsequences $\{\tau'_j\}$, $\{\tau''_j\}$, and numbers $a<b$ such that $\tau'_j\downarrow 0$,

$$\tau''_j\downarrow 0,\quad m(\varphi_{\tau'_j})\to a,\quad m(\varphi_{\tau''_j})\to b.$$

Set $m(f)=\alpha$, $(a+b)/2=\beta$, $\overline{m}=m-m_{\alpha,\beta}$. Then $\overline{m}\in M$ and

$$\overline{m}(f_{\tau'_j})=m(f)+\tau'_j\overline{m}(\varphi_{\tau'_j})=\tfrac{1}{2}(a-b)\tau'_j+o(\tau'_j);\quad \overline{m}(f_{\tau''_j})=m(f)+\tau''_j\overline{m}(\varphi_{\tau''_j})$$

$$=\tfrac{1}{2}(b-a)\tau''_j+o(\tau''_j).$$

This means that the continuous function $\overline{m}(f_x)$ changes sign in every neighborhood to the right of zero. Consequently, the point $x=0$ is a limit point of its zeros. Since, obviously, $\overline{m}(f_x)\not\equiv 0$, this contradicts the A-quasianalyticity of the function f.

By the weak completeness of L, the sequence $\{\varphi_\tau\}$ as $\tau\downarrow 0$ has a weak limit $\varphi_{0+}\in L$. Considering the functions $\varphi_\tau-\varphi_{0+}$, which converge weakly to 0, and using arguments analogous to those given above, we obtain that, as $\tau\downarrow 0$, $\|\varphi_\tau-\varphi_{0+}\|_L\to 0$. In exactly the same way one establishes the existence of a function $\varphi_{0-}\in L$ for which $\|\varphi_\tau-\varphi_{0-}\|_L\to 0$ as $\tau\uparrow 0$. Obviously, the Fourier transforms of the functions φ_{0+} and φ_{0-} coincide. Consequently, φ_{0+} and φ_{0-} coincide almost everywhere. Therefore

$$\lim_{\tau\to 0}\|\varphi_\tau-\varphi_{0+}\|_L=\lim_{\tau\to 0}\|(f_\tau-f)/\tau-\varphi_{0+}\|_L=0,\quad \text{i.e. } f'=\varphi_{0+}.$$

3. Proof of Theorem 2. First of all, conditions 1°–4° are sufficient, since from 1°–4° and ([2]), p. 52, Lemma 1; p. 38, Theorem 1, it follows that for every $m\in M$ the function $\mu(x)=m(f_x)$ belongs to the quasianalytic class of functions (1); consequently, its zeros have no limit point, i.e. f is an A-quasianalytic function.

The necessity of condition 3^0 follows from Theorem 1. Let us prove the necessity of 4^0. Suppose

$$\int_1^\infty \frac{\log F(x)}{x^2}\,dx>-\infty.$$

Then, using 1^0, one can show that if $\nu>0$, $x(\nu)=-F'(\nu)\nu/F(\nu)>0$, then

$$x(\nu) \uparrow \infty, \qquad \min_{\nu \le t \le \nu+1/2} F(t) t^{x(\nu)} \ge \frac{1}{2} F(\nu) \nu^{x(\nu)}. \tag{2}$$

Moreover, if $x \ge 1$,

$$v(x) = \max_{r \ge x} r^{x-[\ln x]} F(r), \qquad c > 1, \qquad \psi_c(r) = \max_{r \ge x \ge 1} \frac{(cr)^x}{v(x)},$$

then

$$\int_1^\infty \frac{\ln \psi_c(r)}{r^2}\, dr < \infty. \tag{3}$$

Put $v(x) = 1$, $0 \le x < 1$, and choose λ_n so that $\{\lambda_n\}$ is increasing and the λ_n run through all values from the set of numbers $2, 4, 6, \dots, 2n, \dots;\ 1^2, 3^2, 5^2, \dots, (2n+1)^2, \dots$. Then, for $r \ge 1$ and some $D > 0$,

$$\theta(r) = 2 \sum_{\lambda_n \le r} \frac{1}{\lambda_n} \le \ln Dr, \qquad U(r) = e^{\theta(r)} \le Dr,$$

$$V(r) = \sup_{0 \le x \le r} \frac{[U(r)]^x}{v(x)} = O(\psi_D(r)),$$

and, by virtue of (3),

$$\int_1^\infty \frac{\ln V(r)}{r^2}\, dr < \infty. \tag{4}$$

Next, let $V(r) = V(-r)$ and $V(r) = 1$ for $|r| < 1$. Put

$$g(z) = \frac{2x}{\pi} \int_{-\infty}^{\infty} \frac{\ln V(\tau)\, d\tau}{x^2 + (y-\tau)^2}, \qquad x > 0, \qquad z = x + iy;$$

(z) is the function conjugate to $g(z)$; $h(x) \equiv 0$ for $x > 0$;

$$\omega(z) = e^{-g(z) - ih(z)};$$

$$G(z) = \prod_{n=1}^{\infty} \frac{\lambda_n - z}{\lambda_n + z} e^{2z/\lambda_n}; \qquad \Phi(z) = C^{-(1+z)} (1+z)^{-A} G(z) \omega(1+z),$$

where A and C are sufficiently large positive numbers. From (4) and (5), p. 19, it follows that, for $x \ge 0$, $\Phi(z)$ has the representation

$$\Phi(z) = \int_0^\infty \varphi(u) u^z \, du,$$

where

$$\int_0^\infty |\varphi(u)| u^x \, du = O(v(x)). \tag{5}$$

In particular, setting $\varphi(-u) = \varphi(u)$, we have

$$\operatorname{sgn} \int_{-\infty}^\infty |\varphi(u)| u^x \, du = \operatorname{sgn} G(x). \tag{6}$$

Let us show that $u\varphi(u)/F(u) \in L$. Indeed, the function $r^{xF}(r)$ attains its maximum at r satisfying the equality $x = x(r)$*. Therefore, setting $x_n = x(n/4)$, $n = 1, 2, \dots, \infty$, we have

$$v(x_n+3) = \max_{r \geq x_n+3} r^{x_n+3-[\ln(x_n+3)]} F(r) = O(1) \max_{r>0} r^{x_n} F(r) = O(1)(n/4)^{x_n} F(n/4).$$

* Recall that $x(\nu) = -F'(\nu)\nu/F(\nu)$ and $x'(\nu) \uparrow \infty$.

Moreover, for $n/4 \leqslant u \leqslant (n+1)/4$, by (2), $u^{x_n} F(u) \geqslant \frac{1}{2}(n/4)^{x_n} F(n/4)$. Hence, by (5),

$$\int_{n/4}^{(n+1)/4} u \frac{|\varphi(u)|}{F(u)} \, du = \int_{n/4}^{(n+1)/4} u^{-2} \frac{|\varphi(u)| u^{3+x_n}}{F(u) u^{x_n}} \, du \leqslant$$

$$\leqslant \left(\frac{n}{4}\right)^{-2} \frac{2}{(n/4)^{x_n} F(n/4)} \int_0^\infty |\varphi(u)| u^{x_n+3} \, du = O(1) \frac{\nu(x_n+3)}{(n/4)^{x_n} F(n/4)} = O(n^{-2}).$$

Thus,

$$\int_{-\infty}^\infty |u| \frac{|\varphi(u)|}{F(u)} \, du = O(1) \left[\sum_1^\infty n^{-2} + \int_0^1 |u\varphi(u)| \, du \right] < \infty.$$

Next, put

$$m(x) = \int_{-\infty}^\infty \frac{u\varphi(u)}{F(u)} \sin ux \, du.$$

Then $m(x) \in M$ and

$$m(fx) = \int_{-\infty}^{\infty} u\varphi(u) \sin ux \, du = \Psi(x).$$

We shall show that $x = 0$ is a limit point of the zeros of $\Psi(x)$, and thereby prove the theorem*. For $2n < x < 2(n+1)$ we have

$$\int_0^{\infty} \left(\int_{-\infty}^{\infty} \varphi(u)(1 - \cos u\tau) \, du \right) \tau^{-(x+1)} \, d\tau =$$

$$= \int_0^{\infty} \left(\int_{-\infty}^{\infty} \varphi(u) \left(\sum_{k=0}^{n} (-1)^k \frac{\tau^{2k} u^{2k}}{(2k)!} - \cos u\tau \right) du \right) \tau^{-(x+1)} \, d\tau =$$

$$= \int_{-\infty}^{\infty} \varphi(u) \, du \int_0^{\infty} \left(\sum_{k=0}^{n} (-1)^k \frac{\tau^{2k} u^{2k}}{(2k)!} - \cos u\tau \right) \tau^{-(x+1)} \, d\tau =$$

$$= (-1)^n \frac{\Gamma(x+1)}{\Gamma(x+1-2n)} \int_0^{\infty} \frac{1 - \cos u\tau}{\tau^{x+1-2n}} \, d\tau \int_{-\infty}^{\infty} \varphi(u) |u|^x \, du,$$

since, by (6),

$$\int_{-\infty}^{\infty} \varphi(u) u^{2k} = 0, \qquad k = 1, 2, \dots, \infty.$$

Consequently, if

$$\omega(\tau) = \int_{-\infty}^{\infty} \varphi(u) \cos u\tau \, du,$$

then, by (6), for $2n < x < 2n + 1$,

$$\operatorname{sgn} \int_0^{\infty} \frac{\omega(0) - \omega(\tau)}{\tau^{x+1}} \, d\tau = (-1)^n \operatorname{sgn} G(x).$$

It follows from this and from the definition of $G(x)$ that on each interval

$$(2n+1)^2 - 1 < x < (2n+1)^2 + 1$$

the function

$$R(x) = \int_0^\infty \frac{\omega(0) - \omega(\tau)}{\tau^{x+1}}\, d\tau$$

changes sign. Since $\omega(\tau) = O(1)$, this is possible only if $\omega(\tau) \equiv 0$ in some right-hand neighborhood of 0, or if $\omega(\tau)$ changes sign in every right-hand neighborhood of 0. In both cases $\tau = 0$ must be a limit point of the zeros of $\omega'(\tau)$. But $\omega'(\tau) = \Psi(\tau)$; therefore $\tau = 0$ is a limit point of the zeros of $\Psi(\tau)$.

Leningrad Civil Engineering Institute

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* Obviously, $\Psi(x) \not\equiv 0$.

Note: Figure translations are in progress. See original paper for figures.