Reports of the Academy of Sciences of the USSR

1965

SovietRxiv

Abstract

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Reports of the Academy of Sciences of the USSR

1965, Volume 164, No. 1

MATHEMATICS

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ON SINGULAR INTEGRAL OPERATORS ON A GROUP

(Presented by Academician V. I. Smirnov on February 11, 1965)

We shall consider mainly singular operators on a commutative group. They preserve all the basic properties of singular operators on R^m.* In those cases where the group is a differentiable manifold (an m-dimensional torus, cylinder), a singular operator on the group can be represented as an integral operator whose kernel, up to a bounded summand, coincides in a neighborhood with the kernel of a singular operator on R^m ([1,2]). In addition to operators on a commutative group, we consider "singular" operators on homogeneous spaces connected with the Lorentz group. These operators lose many important properties of singular operators on R^m. In particular, such generalized operators, generally speaking, are not integral operators with a kernel whose local behavior coincides with the local behavior of a singular kernel in Euclidean space.

1. Let G be a commutative locally compact nondiscrete group; let X be its character group, endowed with the topology of uniform convergence on compact subsets. The group X is also a locally compact group ([3]).

The group G, generally speaking, is noncompact; by adding one "infinitely remote" point it can be made into a compact space $((^4), p.119)$, which we shall denote by G'. Since G is nondiscrete, the group X is noncompact ([5]). The corresponding compactification will be denoted by X'.

Suppose that the group X has no elements of finite order. Let $\Phi(\lambda\chi)$ be a homogeneous function of degree zero, defined on the set $\hat{X}$ of formal products $\lambda\chi$, where $\chi \in X$, and λ is a real number.

Definition 1. The restriction $\Phi(\chi)$ of the function $\Phi(\chi)$ to the group X will be called a homogeneous function of degree zero on the group X.

Definition 2. A singular integral operator with symbol $\Phi(\chi)$ (independent of the pole) on the group G is an operator of the form

$$Au = F^{-1}\Phi(\chi)Fu,$$

where F denotes the Fourier transform on the group G.

The following theorem is a generalization of the theorem of S. G. Mikhlin ([6]).

Theorem 1. *Suppose that the infinitely remote point of the compactification X' has a countable fundamental system of neighborhoods. Let the symbol $\Phi(\chi)$ of the operator A have the following properties: 1) $|\Phi(\chi)| \leq \text{const}$; 2) for every closed neighborhood $W_\infty \subset X'$ of the infinitely remote point not containing zero, and for every positive number ε, there exists a neighborhood of zero W_0 such that, for $\chi \in W_\infty$, $\tau \in W_0$, the inequality*

$$|\Phi(\chi - \tau) - \Phi(\chi)| < \varepsilon; \tag{1}$$

* R^m is m-dimensional Euclidean space.

3) For every closed neighborhood of zero $V_0 \subset X$, not containing the point at infinity, and for every positive number ε, there exists a neighborhood V_∞ of the point at infinity such that, for $\chi \in V_0$, $\tau \in V_0$, inequality (1) is satisfied.

Finally, let the function $a(g)$ be continuous on the compactum G'. Then the operator $(aA - Aa)$ is completely continuous in the space $L^2(G)$.

We now consider operators of the form

$$\sum_k a_k(g)A_k u + Tu, \tag{2}$$

where $a_k(g)$ are functions continuous on G'; A_k are singular operators whose symbols $\Phi_k(\chi)$ satisfy conditions 1)–3) of Theorem 1; T is a completely continuous operator in $L^2(G)$. The sum is assumed to be finite. Denote by S the closure (in the operator norm) of the set of operators of the form (2), and by V the set of completely continuous operators. It follows from Theorem 1 that S is a normed noncommutative ring, and S/V is a commutative ring. Operators from the ring S will be called singular operators.

Theorem 2*. Let the zero of the group G and the point at infinity of the compactum X' have countable fundamental systems of neighborhoods. If the functions $a_k(g)$ are continuous on G', and the symbols $\Phi_k(\chi)$ of the operators A_k satisfy the conditions of Theorem 1, then the inequality

$$\sup_{g,\chi} \left| \sum_k a_k(g)\Phi_k(\chi) \right| \leq \inf_{T\in V} \|A + T\|,$$

where

$$A = \sum_k a_k(g) A_k.$$

This theorem, just as in (7), makes it possible to define the symbol of a singular operator from S. Namely, the symbol of the operator $\sum a_k(g)A_k + T$ will be called the function $\sum a_k(g)\Phi_k(\chi)$. An arbitrary operator A from S is the limit of a sequence of operators of the form (2), and the corresponding sequence of symbols converges uniformly to a certain function $\Phi(g,\chi)$. This function will be called the symbol of the operator A.

From Theorems 1 and 2, in the usual way ([6],[7]), the following is obtained.

Theorem 3. Suppose that the conditions of Theorem 2 are satisfied. Then, in order that a singular operator $A \in S$ be a Φ-operator, it is necessary and sufficient that $\inf|\Phi(g,\chi)| > 0$, where $\Phi(g,\chi)$ is the symbol of the operator A.

2. Let H be the Hilbert space of functions $f(z_1, z_2)$, defined on the complex affine plane, with norm

$$\|f\|^2 = \left(\frac{i}{2}\right)^2 \int |f(z_1,z_2)|^2 \, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2.$$

The role of the Fourier transform for functions $f \in H$ is played by the following Mellin transform ([8]):

$$F(z_1, z_2; \rho, n) = \frac{i}{2} \int f(\lambda z_1, \lambda z_2) \lambda^{-n_1} \bar{\lambda}^{-n_2} d\lambda \wedge d\bar{\lambda}, \qquad n_{1,2} = \frac{\pm n + i\rho}{2},$$

where n is an integer and $\rho \in (-\infty, +\infty)$. The inversion formula holds

$$f(z_1, z_2) = \frac{1}{4\pi^2} \sum_n \int_{-\infty}^{+\infty} F(z_1, z_2; \rho, n) \, d\rho.$$

* This theorem generalizes a theorem of I. Ts. Gokhberg from (7).

We note that the pair (ρ, n) is an element of the group $R \times N$ (here N is the group of integers). Let $\Phi(\rho, n)$ be a homogeneous function of degree zero on the group $R \times N$. By analogy with the definition of a singular operator of convolution type on R^m, we introduce the following

Definition 3. A **singular integral operator on the complex affine plane with symbol** $\Phi(\rho, n)$**, independent of the pole**, is an operator of the form

$$Af = \frac{1}{4\pi^2} \sum_n \int_{-\infty}^{+\infty} \Phi(\rho, n) F(z_1, z_2; \rho, n)\, d\rho. \tag{3}$$

On the complex affine plane we make the change of variables $z_1 = e^{\sigma \pm i\tau} z,\ z_2 = e^{\sigma + i\tau}$ and obtain formula (8)

$$\|f\|^2 = \frac{i}{2} \int |f_z(\sigma, \tau)|^2\, d\sigma \wedge d\tau \wedge dz \wedge d\bar{z}, \tag{4}$$

where by $f_z(\sigma, \tau)$ we denote the function $e^{2\sigma} f(e^{\sigma + i\tau} z,\ e^{\sigma + i\tau})$. After the indicated change of variables, the operator (3) assumes the form

$$(Af)_z(\sigma, \tau) = F^{-1} \Phi(\rho, n) F f_z(\sigma, \tau), \tag{5}$$

where Ff_z is the Fourier transform (in σ and τ) of the function $f_z(\sigma, \tau)$,

$$Ff_z = \int_{-\infty}^{+\infty} \int_0^{2\pi} f_z(\sigma, \tau) e^{-i(\rho\sigma + n\tau)}\, d\sigma\, d\tau.$$

Let $L^2(\sigma, \tau)$ denote the space of square-summable functions $\varphi(\sigma, \tau)$. Formula (4) makes it possible to regard the space H of functions $f(z_1, z_2)$ as a space of abstract functions $f_z(\sigma, \tau)$, specified on the z-plane, with values in $L^2(\sigma, \tau)$.

Now let $a(z_2) = a(\sigma, \tau)$ be a function continuous on the z_2-plane completed by the infinitely distant point, and let the symbol $\Phi(\rho, n)$ satisfy the conditions of Theorem 1. Then the operator of multiplication by $a(z_2)$ and an operator of the form (5) commute up to an operator $\mathcal{T}$ completely continuous in the space $L^2(\sigma, \tau)$. As in Sec. 1, one can introduce the ring S_{z_2} of singular operators with symbols $\Phi(z_2; \rho, n)$.

Theorem 4. *Let the symbol Φ of an operator A from the ring S_{z_2} be such that the inequality $\inf |\Phi| > 0$ holds.*

Then, in order that the equation $Au = v,\ v \in H$ be solvable in the space H, it is necessary and sufficient that the "curve" $v_z(\sigma, \tau)$ lie in the subspace $L^2(\sigma, \tau)$ orthogonal to only a finite number of orths $l_1, \dots, l_N$ of the space $L^2(\sigma, \tau)$. For solvability of the adjoint equation $A^* g = f,\ f \in H$, it is necessary and sufficient that the "curve" $f_z(\sigma, \tau)$ lie in the subspace $L^2(\sigma, \tau)$ orthogonal to the same number of orths $m_1, \dots, m_N$ of the space $L^2(\sigma, \tau)$.

Similarly one can consider the ring S_{z_1}.

We note that the singular operator on the cone K with symbol independent of the pole, obtained as a result of a similar generalization, has the form

$$Af(x) = af(x) + b\int_0^{\infty} \frac{f(t^2x)\,t\,dt}{\ln t}, \qquad x \in K,$$

where a and b are constants.

Finally, all the operators considered whose symbols are independent of the pole had the property that their symbols depended only on the series of the representation (8). In Lobachevsky space, in view of the symmetry relations (8), only the operators of multiplication by a constant have this property.

3. We state a theorem on the boundedness of a singular operator with periodic kernel in $L^p(G)$,

$$G = \{(x_1, \dots, x_m),\ |x_i| \le 1/2;\ i = 1, \dots, k;\ -\infty < x_j < +\infty;$$

$$j = k+1, \dots, m;\ k < m\},$$

which is an exact analogue of the Calderon–Zygmund theorem [(9)] on the boundedness of a singular operator in $L^p(R^m)$.

Theorem 5. Let the periodic kernel $L(x, x-y)$ (2) be generated by the singular kernel

$$K(x, x-y) = \frac{f(x,\theta)}{r^m}, \qquad r = |x-y|, \qquad \theta = \frac{y-x}{r}$$

(the function $f(x,\theta)$ is periodic in the first k coordinates of the point x).

Suppose that the following conditions are satisfied:

1) For every $x \in G$ the series

$$\sum_n{}' K(x, x_1 - y_1 - n_1, \dots, x_k - y_k - n_k;\ x_{k+1} - y_{k+1}, \dots, x_m - y_m)$$

converges almost everywhere to a locally integrable function, and, moreover, the partial sums of the series possess a locally integrable majorant. (The prime on the summation sign means that the summation is carried out over sets of numbers $(n_1, \dots, n_k)$, each of which contains at least one number n_i such that $|n_i| \ge 2$.)

2)

$$\int_S |f(x,\theta)|^{p'}\, dS_\theta \le c = \text{const},$$

where S is the $(m-1)$-dimensional unit sphere.

Then the singular operator with kernel $L(x, x-y)$ is bounded in the space

$$L^p(G), \qquad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 < p < \infty.$$

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Received
8 II 1965

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Note: Figure translations are in progress. See original paper for figures.