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Abstract

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MATHEMATICS

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COMMUTATIVE SYSTEMS AS GENERALIZED SEMIGROUPS

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In this note, by a semigroup we shall always mean a semigroup with identity. We shall denote it by S^1, the set of all its left translations by F, and the right translations by G. Further, let us denote the set of all elements of S^1 by the letter X. Obviously, F and G are anti-isomorphic semigroups of transformations with respect to the operation of composition of transformations. We denote this operation by $\circ$. F and G have the following property: if $f : X \to X$, $g \circ f = f \circ g$ for all $g \in G$, then $f \in F$, and if $g : X \to X$, $g \circ f = f \circ g$ for all $f \in F$, then $g \in G$.

The present note has as its aim the study of the following concept.

Definition. Let X be a set, and let F and G be systems of transformations of X. The triple (F, G, X) is called a **commutative system**, or a k-system, if

$$f : X \to X, \quad f \circ g = g \circ f \quad \text{for all } g \in G \text{ implies } f \in F,$$

$$g : X \to X, \quad f \circ g = g \circ f \quad \text{for all } f \in F \text{ implies } g \in G.$$

A k-system (F, G, X) is called **generated by a semigroup** (group) if it is possible to define on X a semigroup (group) operation for which F is the system of all left translations and G the system of all right translations.

Obviously, every semigroup uniquely determines a k-system, where F is the set of all left translations and G the set of all right translations. On the other hand, not every k-system can in general be obtained in the manner described. A remarkable property of k-systems is that they possess many properties analogous to properties of semigroups. This makes it possible to apply some methods of the study of semigroups also to k-systems. We shall show that every family of transformations of any set can be embedded in a natural way into a k-system. This makes it possible to study families of transformations by the methods of semigroup theory and, for example, to obtain theorems on homomorphisms of families of transformations into matrices whose elements are groups with zero. We are able to define the so-called Schützenberger groups even for a single transformation.

Theorem 1. *A k-system (F, G, X) is generated by a semigroup if and only if there exists $e \in X$ such that $F(e) = G(e) = X$. A k-system is generated by a group if and only if $F(x) = G(x) = X$ for all $x \in X$.*

Obviously, if (F, G, X) is a k-system, then (G, F, X) is also a k-system. This makes it possible to introduce the principle of duality: if a theorem is true for the k-system (F, G, X), then it is true after replacing all concepts by the concepts dual to them for the k-system (G, F, X). In what follows we always assume that (F, G, X) is a k-system. We define for it the following relations on X:

$$xLy \text{ if and only if } F(x) = F(y),$$

$$xRy \text{ if and only if } G(x) = G(y),$$

$$H = R \cap L, \quad D = R \circ L,$$

where $\circ$ denotes composition of relations. Obviously,

L, R, H are equivalences. The class containing x in the equivalence H will be denoted by H_x.

Lemma 1. $R \circ L \subset L \circ R$.

Using duality, we obtain $L \circ R \subset R \circ L$ and the following theorem.

Theorem 2. $R \circ L = L \circ R$. Therefore D is a symmetric relation. Furthermore, D is a transitive relation.

Lemma 2. Let $g_1(x) = y,\ g_2(y) = x,\ g_1, g_2 \in G$, i.e. xRy. Then:

1) $g_1 \circ g_2|F(y) = i|F(y), \quad g_2 \circ g_1|F(x) = i|F(x)$, where $g|Y$ denotes g on Y, and i is the identity mapping;
2) g_1 maps $F(x)$ bijectively onto $F(y)$, and g_2 maps $F(y)$ bijectively onto $F(x)$;
3) $g_1(H_x) \subset H_y$;
4) $g_1|H_x$ is a bijective mapping of H_x onto H_y, and $g_2|H_y$ is a bijective mapping of H_y onto H_x.

Definition. Let $\varphi : X \to Y,\ \bar{X} \subset X$. By $\varphi \parallel \bar{X}$ we denote the mapping defined for all $x \in \bar{X}$ such that $\varphi(x) \in \bar{X}$, and for them

$$\varphi \parallel \bar{X}(x) = \varphi(x).$$

If $\varphi(x) \notin \bar{X}$ for all $x \in X$, then $\varphi \parallel \bar{X}$ loses its meaning.

Corollary. If $g(H_x) \cap H_x \neq \emptyset$, then $g \parallel H_x = g|H_x$. $G \parallel H_x = \{g \parallel H_x \mid g \in G\}$ contains only bijective mappings of H_x onto H_x. Dually, $F \parallel H_x$ contains only bijective mappings of H_x onto H_x.

Theorem 3. Let xHy. Then there exist $\bar{f} \in F \parallel H_x,\ \bar{g} \in G \parallel H_x$ such that

$$\bar{f}(x) = y, \quad \bar{g}(x) = y.$$

By Theorem 1 we obtain:

Theorem 4. For every $x \in X$, $(F \parallel H_x,\ G \parallel H_x,\ H_x)$ is a κ-system generated by a group. $F \parallel H_x$ and $G \parallel H_x$ are anti-isomorphic groups.

Theorem 5. If xRy, then $F \parallel H_x$ and $F \parallel H_y$ are isomorphic groups.

Theorem 6. If xDy, then $F \parallel H_x$ and $F \parallel H_y$ are isomorphic groups.

The groups $F \parallel H_x$ will be called **Schützenberger groups**, or simply **Sch-groups** for F.

Definition. Let Φ be a semigroup of transformations of X, $Y \subset X$, $0 \notin X$, $\varphi \in \Phi$. By $\varphi \parallel Y$ we denote the transformation of $Y \cup \{0\}$ defined as follows:

$$\varphi \parallel Y(0) = 0, \quad \varphi \parallel Y(x) = \varphi(x)$$

for all $x \in Y$ such that $\varphi(x) \in Y$;

$$\varphi \parallel Y(x) = 0$$

for all $x \in X$ such that $\varphi(x) \notin Y$. $\Phi \parallel Y$ denotes the system $\{\varphi \parallel Y \mid \varphi \in \Phi\}$. The mapping σ of the system Φ onto $\Phi \parallel Y$, defined by the equality

$$\sigma(\varphi) = \varphi \parallel Y,$$

will be called the **natural mapping**.

Theorem 7. σ is a homomorphic mapping of Φ onto $\Phi \parallel Y$ if and only if

$$\varphi(x) \notin Y \Rightarrow \psi[\varphi(x)] \notin Y$$

for all $\varphi, \psi \in \Phi$.

Let $z \in X$, and let R_z be the class of all elements equivalent with respect to R. Denote by $\{H_\alpha,\ \alpha \in A\}$ the system of all equivalence classes with respect to H that are contained in R_z. In each H_α we choose exactly one element $x_\alpha \in H_\alpha$. For all $\alpha, \beta \in A$ there exist mappings $g^*_{\alpha,\beta} \in G$ such that

$$g^*_{\alpha,\beta}(x_\beta) = x_\alpha.$$

Put

$$g_{\alpha,\beta} = g^*_{\alpha,\beta}|H_\beta.$$

Theorem 8. $g_{\alpha,\beta}$ is a bijective mapping of H_β onto H_α,

$$g_{\alpha,\alpha} = i|H_\alpha, \quad g_{\alpha,\beta} \circ g_{\beta,\gamma} = g_{\alpha,\gamma}$$

for all $\alpha, \beta, \gamma \in A$.

Lemma 3. The natural mapping G into $G \parallel R_z$ is a homomorphism for all $z \in X$.

But for $G \parallel R_z$ one can naturally construct an isomorphism onto a semigroup of matrices whose elements are from a Sch-group or are zero. To show clearly this idea of Schützenberger homomorphisms, we first examine a special case. By Lemma 2, all H_α have the same cardinality. Let $H_\alpha = \{x_\alpha\}$, i.e. H_α contains only one element. Then all $G \parallel H_\alpha$ are trivial groups. Let $g \in G$. We denote by $M(g)$ the following matrix:

$$M(g) = \{a_{\alpha,\beta}\},$$

where $\alpha, \beta \in A$ and $a_{\alpha,\beta} = 1$ if $g(x_\beta) = x_\alpha$; $a_{\alpha,\beta} = 0$ if $g(x_\beta) \neq x_\alpha$. For $\{a_{\alpha,\beta}\}$, $\{b_{\alpha,\beta}\}$ put

$$\{a_{\alpha,\beta}\} \cdot \{b_{\alpha,\beta}\} =$$

$$= \{c_{\alpha,\beta}\}, \quad \text{where } c_{\alpha,\beta} = \sum_{\gamma \in A} a_{\alpha,\gamma} \cdot b_{\gamma,\beta}.$$

Obviously, if $g_1, g_2 \in G$, then $M(g_1) \cdot M(g_2)$ has meaning, since at most one term in the sum can be nonzero.

Theorem 9. *The mapping ρ, $\rho(g) = M(g)$, is an isomorphism of the semigroups $G \parallel R_z$ and $\{M(g) \mid g \in G\}$.*

If H_α contains more than one point, then $G \parallel H_\alpha$ is a nontrivial group. By Theorems 4 and 5, we obtain that $G \parallel H_\alpha$ and $G \parallel H_\beta$ are isomorphic. We choose one element of A, say ω, and put $M(g) = \{a_{\alpha,\beta}\}$, where $a_{\alpha,\beta} = g_{\omega,\alpha} \circ g \circ g_{\beta,\omega}$, if $g(H_\beta) = H_\alpha$; $a_{\alpha,\beta} = 0$, if $g(H_\beta) \neq H_\alpha$.

It is easy to show that Theorem 9 holds even in the general case if we put $a \cdot b = a \circ b$, if $a \neq 0$, $b \neq 0$; $a \cdot b = 0$, if either $a = 0$ or $b = 0$. In the sum $\sum_{\gamma \in A} a_{\alpha,\gamma} \cdot b_{\gamma,\beta}$ there exists at most one $\delta \in A$ such that $a_{\alpha,\delta} \neq 0$, $b_{\delta,\beta} \neq 0$. Put

$$\sum_{\gamma \in A} a_{\alpha,\gamma} \cdot b_{\gamma,\beta} = a_{\alpha,\delta} \cdot b_{\delta,\gamma}.$$

If such a δ does not exist, put the sum equal to zero. The theorem on homomorphisms can now be formulated as follows:

Theorem 10. *Let (F, G, X) be a κ-system. For each $x \in X$ there exists a nontrivial homomorphism of G onto the semigroup of matrices with elements from the Schützenberger group $G \parallel H_x$ and zero. Dually, there exists a nontrivial* *homomorphism of F onto the semigroup of matrices with elements from the Schützenberger group $F \parallel H_x$ and zero. If (F, G, X) is generated by a semigroup, then the homomorphism G is an antihomomorphism F.*

Obviously, in the case of κ-systems generated by a semigroup, we obtain the Schützenberger representation ([1]). A special example of this representation was considered in ([2]) for κ-systems (F, F, X). This case is simple, since $L = R = H$, and the matrix $M(g)$ has only one element. Then Theorem 10 concerns homomorphisms onto a group with zero. In ([2]) the dependence of the groups $F \parallel H_x$ for different x is shown.

We note that the case, so to speak, opposite to κ-systems generated by a semigroup, where the cardinalities of F and G are equal, is $F = \{i\}$, $G = X^X$. But then in the assertion of Theorem 10 the homomorphism can be replaced by an isomorphism.

Let Φ be a family of transformations of X. We put $K(\Phi) = \{g \mid g : X \to X,\ g \circ \varphi = \varphi \circ g$ for all $\varphi \in \Phi\}$. If we put $F = K[K(\Phi)]$, $G = K(\Phi)$, then $\Phi \subset F$, and (F, G, X) is a κ-system. As an example, take any set X, and let Φ be the family of all constant transformations. Then $K(\Phi)$ consists only of the identity transformation, while $K[K(\Phi)]$ contains all transformations of X.

Let Φ be any system of transformations of X. By a Schützenberger group of Φ we shall mean every Schützenberger group $K[K(\Phi)]$ relative to the κ-system $(K[K(\Phi)], K(\Phi), X)$.

Let us say a few words about the Schützenberger groups of a single transformation φ, i.e. $\Phi = \{\varphi\}$. In this case all Schützenberger groups are commutative and are isomorphic either to finite cyclic groups or to the group of integers. In considering Schützenberger groups, the paper ([4]) is useful. The following theorem can be proved.

Theorem 11. *Let φ be a one-to-one transformation of a finite or countable set X. φ is a right shift of some group if and only if all its Schützenberger groups are isomorphic. If φ is a shift of a group, then it is a shift of a commutative group.*

Theorem 12. *Let φ be a one-to-one transformation of a set X. φ is a right shift of a semigroup if and only if there exists a Schützenberger group of φ that can be homomorphically mapped onto any Schützenberger group of φ. If φ is a right shift of a semigroup, then it is a right shift of a commutative semigroup if and only if its*

Π-group onto which any Π-group φ can be homomorphically mapped.

We shall indicate one more application of k-systems. Let X be a finite set, and let (F, Γ, X) be a k-system. Suppose that each $f \in \Gamma$ has a fixed point. It is known that if the k-system (F, F, X) is generated by a semigroup, then this semigroup has a zero. This means that all $f \in F$ have the same fixed point. But the latter assertion can be proved, using Π-groups, for any such k-system

(F, F, X) ([2]). One can also obtain a more general theorem even for infinite X. At first glance, this theorem does not concern algebraic semigroups, since it establishes a correspondence between F and X. But in essence, the proof uses theorems on algebraic semigroups generalized to k-systems.

One more remark. If a k-system (F, G, X) is generated by a semigroup S^1, then the one-sided ideals of S^1 are in one-to-one correspondence with subsets $Y \subset X$ such that $F(Y) \subset Y$ or $G(Y) \subset Y$. With this in mind, for k-systems we can define and consider all those notions of semigroup theory that can be given by means of one-sided ideals. For example, one can define the notion of a completely simple k-system, which, for k-systems generated by semigroups, coincides exactly with the notion of a completely simple semigroup ([3]).

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Note: Figure translations are in progress. See original paper for figures.