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Abstract

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ASYMPTOTIC REPRESENTATION OF THE SPECTRAL MATRIX OF A FOURTH-ORDER DIFFERENTIAL OPERATOR

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In this note the spectral matrix of a fourth-order differential operator L on the half-axis $(0, \infty)$ is found. The operator L is defined by the differential expression

$$l(x) = x^{\mathrm{IV}} + q(t)x \tag{1}$$

and by the boundary conditions

$$x(0)\sin\alpha_1 + x'''(0)\cos\alpha_1 = 0,$$

$$x'(0)\sin\alpha_2 + x''(0)\cos\alpha_2 = 0, \tag{2}$$

where $q(t) \geq 0$, $\int_0^\infty q(t)\,dt < \infty$, and α_1, α_2 are real numbers. This problem, more general than that considered by the author in (^{1}), is also solved by the method used in (^{2}). In this general case the investigation is carried out in greater detail, and the resulting transcendental equation for the eigenvalues is solved in a more natural way.

For the spectral matrix of the stated problem one can formulate the following theorem:

Theorem. For the spectral matrix $\rho = \|\rho_{ij}\|$, $i, j = 1, 2$, of the operator L defined above, the following asymptotic formulas hold:

$$\rho_{11}(\tilde{s}_2) - \rho_{11}(\tilde{s}_1) = \frac{4}{\pi h_1^2}(\tilde{s}_2 - \tilde{s}_1)(1 + o(1)),$$

$$\rho_{12}(\tilde{s}_2) - \rho_{12}(\tilde{s}_1) = \rho_{21}(\tilde{s}_2) - \rho_{21}(\tilde{s}_1) = -\frac{4}{\pi h_1 h_2}\left(\frac{\tilde{s}_2^2 - \tilde{s}_1^2}{2} - \frac{\tilde{s}_2 - \tilde{s}_1}{h_2}\right)(1 + o(1)),$$

$$\rho_{22}(\tilde{s}_2)-\rho_{22}(\tilde{s}_1)=\frac{4}{\pi h_2^2}\left[\frac{\tilde{s}_2^3-\tilde{s}_1^3}{3}-\frac{\tilde{s}_2^2-\tilde{s}_1^2}{h_2}+\frac{4(\tilde{s}_2-\tilde{s}_1)}{h_2^2}\right](1+o(1)).$$

To solve the problem, we first consider the boundary-value problem on the finite interval $(0,l)$, defined by the differential equation

$$x^{\mathrm{IV}}+q(t)x=\lambda x, \tag{3}$$

the boundary conditions (2), and the boundary conditions

$$x(l)\sin\beta_1+x'''(l)\cos\beta_1=0,$$

$$x'(l)\sin\beta_2+x''(l)\cos\beta_2=0, \tag{4}$$

where β_1,β_2 are real numbers and λ is a sufficiently large parameter.

Linearly independent solutions of equation (3) are found by the method presented in (^{3}). Next we seek solutions $w_1(t,\lambda)$ and $w_2(t,\lambda)$ of equation (3), satisfying the initial conditions

$$w_1(0,\lambda)=h_1,\quad w_1'(0,\lambda)=0,\quad w_1''(0,\lambda)=0,\quad w_1'''(0,\lambda)=1;$$

$$w_2(0,\lambda)=0,\quad w_2'(0,\lambda)=h_2,\quad w_2''(0,\lambda)=1,\quad w_2'''(0,\lambda)=0,$$

where $h_1=-\operatorname{ctg}\alpha_1,\ h_2=-\operatorname{ctg}\alpha_2$.

Obviously, the linear combination $C_1w_1(t,\lambda)+C_2w_2(t,\lambda)$ satisfies the boundary conditions (2). Therefore, from the requirement that it also satisfy the boundary conditions (4), we find the transcendental equation for determining the eigenvalues of the given boundary-value problem, i.e.

$$\left(p_1+\frac{p_2}{s}\right)\cos sl+\left(\frac{p_2}{s}+\frac{g_1}{s^2}\right)\sin sl+o\left(\frac{1}{s^2}\right)=0, \tag{5}$$

where $p_1=H_1H_2h_1h_2,\ p_2=H_1h_1(H_2+h_2),\ g_1=2H_1h_1$. Here $H_1=\operatorname{ctg}\beta_1$;

$$H_2=\operatorname{ctg}\beta_2;$$

h_1,h_2 are the notation mentioned above; $s=\sqrt[4]{\lambda};\ \lambda>0$.

Obviously, for large s the roots of equation (5) are close to the numbers $(2n+1)\pi/2$. It can be shown that they are simple.

We find the roots of equation (5) by putting

$$s_n l = (2n+1)\pi/2 + \delta_n, \qquad n = 0, 1, 2, 3, \dots,$$

where δ_n is a sufficiently small number. Then we obtain that the difference between two consecutive eigenvalues is

$$\Delta s_n = s_{n+1} - s_n = \frac{\pi}{l}(1 + o(1)).$$

We seek the eigenfunctions in the form of the linear combination

$$C_1 w_1(t, \lambda_n) + C_2 w_2(t, \lambda_n).$$

In determining C_1 and C_2 we use the definition of an eigenfunction. After this we obtain the asymptotic formula

$$w_n(t) = C\left[\frac{h_2}{s_n}(\cos s_n t - \sin s_n t) + \frac{2}{s_n^2}\cos s_n t + o\left(\frac{1}{s_n^2}\right)\right], \tag{6}$$

where $C = -\frac{1}{2}C_2$. Using the way in which we found the solutions $w_1(t, \lambda)$, $w_2(t, \lambda)$ and the eigenfunctions themselves, it is easy to prove the boundedness of $w_n(t)$ for sufficiently large s.

From the normalization condition for the eigenfunctions we have

$$C = \frac{1}{a_n}, \qquad a_n^2 = \int_0^l \widetilde{w}_n^2(t)\,dt,$$

where the function $\widetilde{w}_n(t)$ is defined by equality (6) with $C = 1$.

Since the boundary-value problem (3)–(2)–(4) is self-adjoint and has simple eigenvalues, the system of its eigenfunctions is complete and orthonormal (we can always perform the normalization). Therefore we can write Parseval' s equality for the finite interval. If $f(t) \in L_2(0, \infty)$, then

$$\int_0^l f^2(t)\,dt = \sum_n \left\{C_1^2(\lambda_n)F_1^2(\lambda_n) + 2C_1(\lambda_n)C_2(\lambda_n)F_1(\lambda_n)F_2(\lambda_n) + C_2^2(\lambda_n)F_2^2(\lambda_n)\right\}, \tag{7}$$

where

$$F_k(\lambda) = \int_0^{\xi} f(t) w_k(t,\lambda)\,dt, \qquad k = 1, 2.$$

Finding C_1 and C_2 from equality (7), we obtain the following asymptotic formulas for the spectral matrix in the case of a finite interval:

$$\rho_{11}^{(l)}(s) = \sum_{\tilde{s}1 < s_n \le \tilde{s}_2}^{n} \frac{4}{h_1^2 l}\,(1 + o(1)),$$

$$\rho_{12}^{(l)}(s) = \rho_{21}^{(l)}(s) = \sum_{\tilde{s}1 < s_n \le \tilde{s}_2}^{n} \frac{4}{h_1 h_2 l}\left(s_n - \frac{1}{h_2} + o(1)\right),$$

$$\rho_{22}^{(l)}(s) = \sum_{\tilde{s}1 < s_n \le \tilde{s}_2}^{n} \frac{4}{h_2^2 l}\left(s_n^2 - \frac{2s_n}{h_2} + \frac{4}{h_2^2} + o(1)\right).$$

Multiplying and dividing the right-hand sides of the last equalities by Δs_n and passing in them to the limit as $l \to \infty$ (i.e. $\Delta s_n \to 0$), we obtain the asymptotic formulas of the theorem. In doing so, we use the rules for integrating asymptotic formulas (see at the end of book (^{4})).

In this case it is assumed that $\sin\alpha_k \sin\beta_k \neq 0,\ k = 1, 2$.

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Note: Figure translations are in progress. See original paper for figures.

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