# MATHEMATICS

1965

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## Abstract

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# MATHEMATICS


## V. V. PETROV


## ONE ESTIMATE OF THE DEVIATION OF THE DISTRIBUTION OF A SUM OF INDEPENDENT RANDOM VARIABLES FROM THE NORMAL LAW


*(Presented by Academician Yu. V. Linnik on 18 IX 1964)*


**1.** Let $X_1, X_2, \dots, X_n$ be mutually independent random variables having, generally speaking, different distributions with finite variances, not all of which are equal to zero. Let $EX_j = 0$ $(j = 1, \dots, n)$. Introduce the following notation:

$$\sigma_j^2 = E(X_j^2), \qquad s_n^2 = \sum_{j=1}^{n} \sigma_j^2, \qquad Z_n = \frac{1}{s_n} \sum_{j=1}^{n} X_j,$$

$$F_n(x) = P(Z_n < x), \qquad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

**2.** Lyapunov ([1]) showed that if the additional condition $E|X_j|^{2+\delta} < \infty$ $(j = 1, \dots, n)$ is satisfied for some positive $\delta \le 1$, then the estimate

$$\sup_x |F_n(x) - \Phi(x)| \le \frac{C}{s_n^{2+\delta}} \sum_{j=1}^{n} E|X_j|^{2+\delta}, \tag{1}$$

holds, where $C$ is a constant independent of $n$. For $\delta = 1$, estimate (1) with an absolute constant $C$ was obtained in the case of identical distributions by Berry ([2]) and Esseen ([3]), and in the general case considered here by Esseen ([3,4]). Recently Katz ([5]) obtained, for a sequence of identically distributed random variables, an interesting generalization of the Berry—Esseen theorem.

The present note contains one theorem from which all the above-mentioned results follow as special cases. In its formulation and proof this theorem is close to the work of Katz ([5]), which served as the occasion for writing this note.

**3.** Following ([5]), denote by $G$ the class of functions $g(x)$, defined for all real values of $x$ and satisfying the following conditions:

(A) $g(x)$ is a nonnegative, even function, nondecreasing on $(0, +\infty)$, such that

$$\lim_{x \to +\infty} g(x) = +\infty.$$

(B) The function $x/g(x)$ is defined for all real $x$ and is nondecreasing on $(0,+\infty)$.

**Theorem.** Let $g(x) \in G$. Let $n$ be any positive integer and let $X_1,\dots,X_n$ be random variables satisfying the conditions of item 1 and, in addition, the condition

$$E[X_j^2 g(X_j)] < \infty \qquad (j = 1,\dots,n). \tag{2}$$

*Then*

$$\sup_x |F_n(x) - \Phi(x)| \leqslant \frac{C}{s_n^2 g(s_n)} \sum_{j=1}^{n} E[X_j^2 g(X_j)], \tag{3}$$

where $C$ is some absolute constant.

4. **Proof.** For $j = 1,\dots,n$, put

$$\overline{X}_j = \begin{cases} X_j, & \text{if } |X_j| \leqslant s_n, \\ 0, & \text{if } |X_j| > s_n, \end{cases}$$

$$\overline{a}_j = E\overline{X}_j, \qquad \overline{\sigma}_j^2 = E(\overline{X}_j^2) - (E\overline{X}_j)^2, \qquad \overline{s}_n^2 = \sum_{j=1}^{n} \overline{\sigma}_j^2.$$

Obviously, $\overline{\sigma}_j^2 \leqslant \sigma_j^2$. Denote by $V_j(x)$ the distribution function of the random variable $X_j$. We have

$$\sigma_j^2 - \overline{\sigma}_j^2 = \int_{|x|>s_n} x^2\, dV_j(x) + \left( \int_{|x|\leqslant s_n} x\, dV_j(x) \right)^2 \leqslant$$

$$\leqslant \frac{2}{g(s_n)} \int_{|x|>s_n} x^2 g(x)\, dV_j(x) \leqslant \frac{2}{g(s_n)} E[X_j^2 g(X_j)]. \tag{4}$$

For any natural number $n$, one of the two conditions is satisfied: either

$$\overline{s}_n \leqslant s_n/2, \tag{5}$$

or

$$\overline{s}_n > s_n/2. \tag{6}$$

If $n$ satisfies condition (5), then

$$1 \leqslant \frac{8}{3s_n^2 g(s_n)} \sum_{j=1}^{n} E[X_j^2 g(X_j)]$$

by virtue of (4), so that for such values of $n$ the estimate (3) with $C = 8/3$ is trivial. Therefore, in what follows we shall consider only those values of $n$ for which condition (6) is satisfied.

Put

$$Y_n = \frac{1}{s_n} \sum_{j=1}^{n} \overline{X}_j, \qquad \overline{Z}_n = \frac{1}{\overline{s}_n} \sum_{j=1}^{n} (\overline{X}_j - \overline{a}_j).$$

The event $Z_n < x$ entails the event

$$(Y_n < x) \cup (|X_1| > s_n) \cup \dots \cup (|X_n| > s_n).$$

Consequently,

$$F_n(x) \leqslant P(Y_n < x) + \sum_{j=1}^{n} P(|X_j| > s_n).$$

In the same way we obtain the stronger inequality

$$|F_n(x) - P(Y_n < x)| \leqslant \sum_{j=1}^{n} P(|X_j| > s_n).$$

Therefore, for all $x$ we have

$$|F_n(x) - \Phi(x)| \leqslant$$

$$\leqslant \sup_x \left| P\left( \overline{Z}_n < \left( x s_n - \sum_{j=1}^{n} \overline{a}_j \right) / \overline{s}_n \right) - \Phi\left( \left( x s_n - \sum_{j=1}^{n} \overline{a}_j \right) / \overline{s}_n \right) \right| +$$

$$+ \sup_x \left| \Phi\left( \left( x s_n - \sum_{j=1}^{n} \overline{a}_j \right) / \overline{s}_n \right) - \Phi(x) \right| + \sum_{j=1}^{n} P(|X_j| > s_n). \qquad (7)$$

Denote the three terms on the right-hand side of inequality (7) by $T_1, T_2$, and $T_3$, respectively. Applying to the sequence of random variables

$\overline{X}_1, \dots, \overline{X}_n$ Esseen' s theorem ([3,4]), we find

$$T_1 \leqslant C_0 \overline{s}_n^{-3} \sum_{j=1}^{n} E|\overline{X}_j - \overline{a}_j|^3. \tag{8}$$

According to Bergström' s work ([6]), in (8) one may take $C_0 = 4.8$. Further,

$$E|\overline{X}_j - \overline{a}_j|^3 \leqslant 4(E|\overline{X}_j|^3 + |\overline{a}_j|^3) \leqslant 8E|\overline{X}_j|^3 =$$

$$= 8 \int_{|x|<s_n} \frac{|x|}{g(x)} x^2 g(x)\, dV_j(x) \leqslant \frac{8s_n}{g(s_n)} E[X_j^2 g(X_j)]. \tag{9}$$

From (8), (9), and (6) it follows that

$$T_1 \leqslant \frac{C_1}{s_n^2 g(s_n)} \sum_{j=1}^{n} E[X_j^2 g(X_j)], \tag{10}$$

where $C_1 = 64C_0$.

It is not difficult to find that

$$T_2 \leqslant \frac{1}{\sqrt{2\pi}} \left( (s_n - \overline{s}_n)/\overline{s}_n + \frac{1}{s_n} \left| \sum_{j=1}^{n} \overline{a}_j \right| \right).$$

Further,

$$|\overline{a}_j| \leqslant \frac{1}{s_{ng}(s_n)} E[X_j^2 g(X_j)].$$

Hence, with the aid of (4) and (6), we obtain

$$T_2 \leqslant \frac{C_2}{s_n^2 g(s_n)} \sum_{j=1}^{n} E[X_j^2 g(X_j)], \tag{11}$$

where $C_2 = 14/3\sqrt{2\pi}$. Chebyshev' s inequality gives the estimate

$$T_3 \leqslant \frac{1}{s_n^2 g(s_n)} \sum_{j=1}^{n} E[X_j^2 g(X_j)]. \tag{12}$$

From (7), (10), (11), and (12), (3) follows. In (3) one may take

$$C = 64C_0 + 14/3\sqrt{2\pi} + 1.$$

5. In the special case of identical distributions, the theorem proved becomes Katz’ s theorem ([5]). If $0 < \delta \leqslant 1$, then the function $g(x) = |x|^\delta$ belongs to the class $G$. With this choice of the function $g(x)$ and $\delta < 1$, estimate (3) coincides, up to the constant in the right-hand side, with Lyapunov’ s estimate (1)*, and for $\delta = 1$ with Esseen’ s estimate.

Estimate (3) may be nontrivial also in those cases where the random variables under consideration have no finite power moments of order higher than the second.

Leningrad State University
named after A. A. Zhdanov

Received
7 IX 1964

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* From the theorem proved it follows that Lyapunov’ s estimate holds with an absolute constant $C$.

*Note: Figure translations are in progress. See original paper for figures.*