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Abstract

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ON CONTINUOUS MEASURES

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General properties are described of the sequential order topology in the Boolean ring $\mathcal{P}$ of all subsets of a given set. Measures are considered that are continuous mappings of a subring $\mathcal{A}$ of $\mathcal{P}$ into a regular Abelian semigroup; conditions are given for the extendability of such measures to the closure of $\mathcal{A}$.

1. Consider the set $\mathcal{P} = \mathcal{P}(E)$ of all subsets of the set E. Define in $\mathcal{P}$ the ring structure in the usual way: for arbitrary sets X and Y in $\mathcal{P}$, the sum $X + Y$ is the symmetric difference of these sets, and the product XY is their intersection. We order the set $\mathcal{P}$ by inclusion. Thus $\mathcal{P}$ is a Boolean ring with identity and a complete Boolean algebra (([1]), Ch. 2, § 1; Ch. 10, §§ 1-3). Define in $\mathcal{P}$ a topology (([2]), Ch. 1, § 7) as the strongest of the topologies in $\mathcal{P}$ for which, for every sequence (X_n), $X_n \in \mathcal{P}$, and every $A \in \mathcal{P}$, convergence of (X_n) to A in order $\left(X_n \xrightarrow{o} A\right.$; ([1]), Ch. 4, § 8) implies convergence of (X_n) to A ($X_n \to A$; ([2]), Ch. 1, § 6, item 4). We shall call the topology thus obtained the *so*-topology (cf. ([3])). The set $\mathcal{P}$, with the ring, order, and topology structures defined on it in the indicated way, will be denoted by $\mathcal{P}_{so} = \mathcal{P}_{so}(E)$.

The following propositions describe the general properties of the topological space $\mathcal{P}_{so}$.

A set $\mathcal{F} \subseteq \mathcal{P}_{so}$ is closed if and only if, for every sequence (X_n), $X_n \in \mathcal{F}$, and every $A \in \mathcal{P}_{so}$, from $X_n \xrightarrow{o} A$ it follows that $A \in \mathcal{F}$ (cf. ([4]), Ch. 1, § 2, item 4).

Theorem 1. For every $A \in \mathcal{P}_{so}$, the translation $Y = A + X$ is a homeomorphism.

Theorem 2. The topological space $\mathcal{P}_{so}$ is separated (Hausdorff) and totally disconnected.

Theorem 3. If E is uncountable, the topological space $\mathcal{P}_{so}(E)$ is noncompact.

Theorem 4. The operations $X + Y$ and XY in $\mathcal{P}_{so}$ are continuous in each variable.

Theorem 5. The closure of a ring $\mathcal{A}$ in $\mathcal{P}_{so}$ coincides with the σ-ring generated by $\mathcal{A}$.

2. Consider a topological Abelian semigroup B with regular topology and zero, and a subring $\mathcal{A}$ of the ring $\mathcal{P}_{so}$ with the topology induced from $\mathcal{P}_{so}$. A mapping m of the ring $\mathcal{A}$ into the semigroup B will be called a measure (on $\mathcal{A}$ with values in B) if:

1) $m(0) = 0$,

2) $m(X + Y) = m(X) + m(Y)$ for all X and Y in $\mathcal{A}$ such that $XY = 0$.

Condition 2) in the definition of a measure is equivalent to the condition

2′) $m(X + Y + XY) + m(XY) = m(X) + m(Y)$ for all X and Y in $\mathcal{A}$.

Measures that are continuous mappings of the space $\mathcal{A}$ into the space B will be called **continuous measures**. Consider, for example, continuous measures with values in the semigroup $B = R_+$ of all positive (≥ 0) numbers.

Theorem 6. *A positive numerical measure is continuous if and only if it is countably additive and bounded.*

Indeed, the continuity of a measure implies its countable additivity. The unboundedness of a measure entails its discontinuity at zero, since in this case there exists a sequence of sets having measures ≥ 1 and converging to 0. If a measure is countably additive and bounded on a σ-ring, then the preimage of a closed numerical set is closed in $\mathfrak{P}_{so}$. Hence the sufficiency of the condition of the theorem follows, since a countably additive bounded measure extends uniquely from the ring to the σ-ring generated by it ([5]), Ch. 3, § 13, Theorem 1.

Let $\overline{\mathcal{A}}$ be the closure of the ring $\mathcal{A}$ in $\mathfrak{P}_{so}$, and let m be a continuous measure on $\mathcal{A}$ with values in the semigroup B. We give conditions for the extendability of the measure m from $\mathcal{A}$ to $\overline{\mathcal{A}}$.

Theorem 7. *For the existence of a continuous measure $\overline{m}$ on $\overline{\mathcal{A}}$ that is an extension of the measure m, it is necessary and sufficient that the limit*

$$\lim_{X \ni \overline{X},\ X \in A} m(X)$$

exist for all $\overline{X} \in \overline{\mathcal{A}}$.

The condition of the theorem is necessary and sufficient for

$$\overline{m}(\overline{X}) = \lim_{X \ni \overline{X},\ X \in A} m(X)$$

to be a continuous mapping of $\overline{\mathcal{A}}$ into B ([2]), Ch. 1, § 6, item 7, Theorem 1). It remains to verify condition 2′) for $\overline{m}$. The validity of this condition follows from the continuity of addition in B, the partial continuity of the operations in $\mathfrak{P}_{so}$, the continuity of $\overline{m}$ on $\overline{\mathcal{A}}$, and the fulfillment of the corresponding equality for $\overline{m}$ on $\mathcal{A}$.

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Note: Figure translations are in progress. See original paper for figures.