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ON THE RIEMANN–HILBERT PROBLEM FOR A HOLOMORPHIC VECTOR

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In this note we consider a boundary-value problem of Riemann–Hilbert type (problem A) for a holomorphic vector. A simple condition will be indicated that ensures the Noether property of problem A. Under this condition a formula will be obtained expressing the index of problem A. For this we use I. N. Vekua' s method of reducing a boundary-value problem to an equivalent system of singular integral equations. We note that the Riemann–Hilbert boundary-value problem for a holomorphic vector (in one special case) was first considered in the work of A. V. Bitsadze ([1]).

Let us first obtain an integral representation for a holomorphic vector. Let a closed Lyapunov surface S bound a domain G of three-dimensional space, homeomorphic to a ball, and let $U(x)$ be the desired vector, holomorphic in the domain G and Hölder-continuous in $G+S$. Denote by ν the exterior normal to the surface S at the point ζ, and consider the Fredholm equation

$$\mu_1(y) - \frac{1}{2\pi}\iint\limits_S \frac{\partial}{\partial\nu}\frac{1}{|y-\zeta|}\,\mu_1(\zeta)\,ds = 2f_1(y) \qquad (y\in S), \tag{1}$$

where $f_1(y)$ are the boundary values of the first component of the vector $U(x)$ on S. It is well known (see ([6])) that for a given function $f_1(y)$ equation (1) always has, and moreover has uniquely, a solution $\mu_1(y)$, Hölder-continuous on S, if $f_1(y)$ is Hölder-continuous.

Now consider the holomorphic vector defined by an integral of Cauchy type ([10])

$$V(x) = \frac{1}{4\pi}\iint\limits_S D'\frac{1}{|x-\zeta|}\,DS_\zeta\,\mu(\zeta) \qquad (x\in G), \tag{2}$$

where the vector $\mu(y)$ has components $[\mu_1(y),0,0,0]$.

Let $W(x)=U(x)-V(x)$. Passing to the limit as $x\to y\in S$, we obtain, by virtue of equation (1), that on the surface S the first component w_1 of the vector W is equal to zero. But the components of a holomorphic vector are harmonic functions. Consequently, $w_1(x)\equiv 0$ in the domain G, so that $W=[0,\widetilde{w}]$, where

$\widetilde{w}$ is a three-component vector satisfying in G the system $\operatorname{div} \widetilde{w} = 0$, $\operatorname{rot} \widetilde{w} = 0$. Therefore $\widetilde{w} = \operatorname{grad} \varphi$, where $\varphi(x)$ is a function harmonic in the domain G, Hölder-continuous together with its first-order derivatives in $G + S$.

Let $g(x, \zeta)$ be the Green function of the domain G for the Neumann problem. From Green' s formula there follows the relation

$$\varphi(x) = \iint\limits_S g(x, \zeta) \frac{\partial \varphi}{\partial \nu} \, ds_\zeta + \text{const},$$

so that $\varphi(x)$ may be sought in the form

$$\varphi(x) = \iint\limits_S g(x, \zeta) \left(\mu_2(\zeta) - \iint\limits_S \mu_2 \, ds \right) ds + \text{const}, \tag{3}$$

where $\mu_2(y)$ is an unknown real function, Hölder-continuous on S. Let $\varphi(x) \equiv$ const in G. Differentiating formula (3) in the normal direction and passing to the limit as $x \to y \in S$, we obtain

$$\mu_2(y) = \iint_S \mu_2 \, ds.$$

Consequently, the density $\mu_2(y)$, up to an additive constant, is uniquely determined by the vector $\operatorname{grad} \varphi$.

Thus every vector holomorphic in the domain G and Hölder-continuous in $G+S$ can be represented in the form

$$U(x) = \frac{1}{4\pi} \iint_S D' \frac{1}{|x - \xi|} D S_\xi [\mu_1(\xi), 0, 0, 0] +$$

$$+ \left[0, \operatorname{grad} \iint_S g(x, \xi) \left(\mu_2(\xi) - \iint_S \mu_2 \, ds \right) ds \right], \tag{4}$$

where the first term is an integral of Cauchy type ([10]), $g(x, \xi)$ is the Green function of the domain G for the Neumann problem, and the function $\mu_1(y)$ is uniquely determined by the vector $U(x)$, while the function $\mu_2(y)$ is determined up to a constant; hence the vector $U(x)$ is trivial if and only if $\mu_1 \equiv 0$, $\mu_2 \equiv$ const.

Let us now consider the following boundary-value problem:

Problem A. Find a vector $U(x)$ with components $[p, u, v, w]$, holomorphic in the domain G, Hölder-continuous in $G + S$, and satisfying on S the boundary condition

$$\lambda_{k1} p + \lambda_{k2} u + \lambda_{k3} v + \lambda_{k4} w = f_k(y) \qquad (k = 1, 2), \tag{5}$$

where $\lambda_{kj}(y)$ $(j = 1, 2, 3, 4)$ and $f_k(y)$ are real functions of the point y of the surface S, Hölder-continuous. Without loss of generality one may assume that

$$(\lambda_{k1})^2 + (\lambda_{k2})^2 + (\lambda_{k3})^2 + (\lambda_{k4})^2 = 1 \qquad (k = 1, 2). \tag{6}$$

Denote by Λ_{kj} the determinant formed from the k-th and j-th columns of the matrix of the boundary condition (5), and let

$$A(y) = \Lambda_{12} + \Lambda_{34}; \qquad B(y) = \Lambda_{13} + \Lambda_{42}; \qquad C(y) = \Lambda_{14} + \Lambda_{23}.$$

The vector $l(y)$ with coordinates $(A(y), B(y), C(y))$ will be called the vector of the boundary condition.

Theorem. Let the domain G be the unit ball, and let the vector of the boundary condition not enter the tangent plane to the sphere S. Then the boundary-value problem A is Noetherian; in particular, the homogeneous problem A^0 has only a finite number k of linearly independent solutions, and for solvability of the nonhomogeneous problem A with right-hand side (5) it is necessary to impose a finite number k' of orthogonality conditions.

Suppose, in addition, that the coefficients of the boundary condition (5) are twice continuously differentiable* with respect to the (Cartesian) coordinates of the point y and that condition (6) holds. Then the integer $k - k' = \varkappa$ (the index of problem A)

$$\varkappa = -1. \tag{7}$$

Proof. First of all, note that for the ball the function $g(x, \xi)$ has the form (see [8])

$$g(x,\xi) = \frac{1}{4\pi}\left\{\frac{1}{|x-\xi|} + \frac{1}{|\xi|\cdot|x-\xi^*|} - \frac{1}{2}\left(|\xi|^2 + |x|^2\right) - \right.$$
$$\left. - \ln\left(1 - x\cdot\xi + |\xi|\cdot|x-\xi^*|\right)\right\},$$

where $\xi^* = \xi/|\xi|^2$ and $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$.

Representing now the sought vector $U(x)$ through the unknown densities $\mu_1(y)$ and $\mu_2(y)$ by formula (4) and passing to the boundary, we obtain, by virtue of

* **Note added in proof.** From the stability of the index under small changes of the coefficients it follows that it suffices to impose the requirement that the coefficients of the boundary condition be Hölder-continuous.

conditions (5), for determining the vector $\mu(\mu_1, \mu_2)$, a system of two singular integral equations

$$H\mu = f \quad \{f = (f_1, f_2)\}, \tag{8}$$

where the operator $H\mu$ is easily written out explicitly.

Let us write the symbolic matrix $\Phi(r)$ of the system (8):

$$\Phi(r) = \left\| \begin{matrix} \lambda_{11}(y) + i\lambda_1 \cdot \rho & \lambda_1 \cdot n + i\lambda_1 \cdot r \\ \lambda_{21}(y) + i\lambda_2 \cdot \rho & \lambda_2 \cdot n + i\lambda_2 \cdot r \end{matrix} \right\|, \tag{9}$$

where r and n are, respectively, the unit tangent vector and the outer normal to the sphere S at the point y, $\rho = n \times r$. The determinant of the matrix (9), $l(y) \cdot (n + ir)$, is nonzero by the hypothesis of the theorem, so that the resulting system of singular integral equations is Noetherian (see (7)). Let us note that a necessary and sufficient condition for solvability of problem A can also be formulated in terms of the adjoint problem A' ([9]).

Let us now consider the vector λ with components $[\lambda_{11}, \lambda_1 \cdot \rho, \lambda_1 \cdot n, \lambda_1 \cdot r]$, determined by the first row of the matrix (9). Choosing ρ, n, r as basis vectors of a new coordinate system with center at the point y, we obtain $(\lambda_1 \cdot \rho)^2 + (\lambda_1 \cdot n)^2 + (\lambda_1 \cdot r)^2 = |\lambda_1|^2 = \lambda_{12}^2 + \lambda_{13}^2 + \lambda_{14}^2$, whence, in view of condition (6), it follows that $|\lambda| = 1$.

Let us compute the index of the system (8).

It is known ([5,4,2]) that the index of the system (8) differs only by a constant factor from the rotation $\sigma(\lambda)$ of the vector λ, and for the sphere this factor is equal to one. A formula for computing $\sigma(\lambda)$ is given in ([2]). By means of elementary transformations of the determinant standing under the integral sign, we obtain

$$\sigma(\lambda) = -\frac{1}{\pi} \iint\limits_S \begin{vmatrix} \frac{\partial}{\partial\theta}\lambda_{11} & \frac{\partial}{\partial\theta}(\lambda_1 \cdot \nu) \\ \frac{\partial}{\partial\varphi}\lambda_{11} & \frac{\partial}{\partial\varphi}(\lambda_1 \cdot \nu) \end{vmatrix} d\theta\, d\varphi.$$

By Green' s formula, $\sigma(\lambda) = 0$.

Among the solutions of the homogeneous system (8) there certainly occurs the vector $(0, a)$, $a = \text{const} \neq 0$. In view of the correspondence between U and μ, determined by formula (4), $\varkappa = -1 + \sigma(\lambda)$, whence formula (7) follows.

Although formula (7) has been proved for a ball, using the arguments given above one can show that it is valid for an arbitrary domain G, homeomorphic to a ball, bounded by a three-times continuously differentiable surface S (θ and φ in this case denote intrinsic coordinates on the surface). Here, of course, it is assumed that the vector of the boundary condition $l(y)$ does not enter the tangent plane to the surface S. The Noetherian property of problem A under this condition was proved in ([9]).

Let us further note that a formula for the index of the problem with boundary condition (5) (even for more general systems of the form $DU = \varepsilon BU$ ([9]), where the values of the parameter ε do not belong to a certain discrete set) can be obtained starting from the accompanying boundary problems A_* and A^* ([3]).

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Note: Figure translations are in progress. See original paper for figures.