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**Abstract**

**Full Text**

**L. P. SHILNIKOV**

## ON ONE CASE OF THE EXISTENCE OF A COUNTABLE SET OF PERIODIC MOTIONS

*(Presented by Academician A. Yu. Ishlinsky, 28 VII 1964)*

Let us consider the system

$$\begin{aligned} \frac{dx}{dt} &= \rho x - \omega y + P(x, y, z), & \frac{dy}{dt} &= \omega x + \rho y + Q(x, y, z), \\ \frac{dz}{dt} &= \lambda z + R(x, y, z), \end{aligned} \quad (1)$$

where  $P, Q$ , and  $R$  are analytic functions that vanish at the origin together with their first derivatives.

Suppose that  $\rho < 0$ ,  $\lambda > 0$ . In this case the equilibrium state  $O(0, 0, 0)$ , according to Poincaré's terminology, is called a saddle-focus. The behavior of integral curves in a neighborhood of the saddle-focus is known<sup>(1,2)</sup> and is characterized first of all by the existence of a two-dimensional separatrix surface  $\pi^+$

$$z = \varphi_3(x, y),$$

on which the  $O^+$ -curves lie, and also by the existence of a one-dimensional integral manifold  $\pi^-$

$$x = \varphi_1(z), \quad y = \varphi_2(z),$$

on which two  $O^-$ -curves lie. The functions  $\varphi_1, \varphi_2$ , and  $\varphi_3$  are analytic and satisfy the conditions

$$\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = \varphi_3(0, 0) = \frac{\partial \varphi_3(0, 0)}{\partial x} = \frac{\partial \varphi_3(0, 0)}{\partial y} = 0.$$

Suppose that one of the trajectories leaving the saddle-focus returns to it as  $t \rightarrow \infty$ . Denote this trajectory by  $\Gamma_0$ .

**Theorem.** *If  $\lambda > -\rho$ , then every neighborhood of the trajectory  $\Gamma_0$  contains a countable set of periodic solutions.*

For the proof of the theorem we shall use the method of point transformations. As in <sup>(3)</sup>, we shall construct a point transformation  $T$ , represented in the form of the product of two transformations  $T_0$  and  $T_1$ , where  $T_0$  is a transformation in a neighborhood of the saddle-focus, and  $T_1$  is a transformation in a neighborhood of  $\Gamma_0$ .

Make the nonlinear change of variables

$$\xi = x - \varphi_1(z), \quad \eta = y - \varphi_2(z), \quad \zeta = z - \varphi_3(x, y),$$

and also a change of time, as a result of which system (1) in a certain  $\varepsilon$ -neighborhood of the origin is reduced to the form:

$$\begin{aligned} \frac{d\xi}{dt} &= \rho\xi - \omega\eta + P_1(\xi, \eta, \zeta)\xi + P_2(\xi, \eta, \zeta)\eta, \\ \frac{d\eta}{dt} &= \omega\xi + \rho\eta + Q_1(\xi, \eta, \zeta)\xi + Q_2(\xi, \eta, \zeta)\eta, \\ \frac{d\zeta}{dt} &= \lambda. \end{aligned} \quad (2)$$

where  $P_1, P_2, Q_1$ , and  $Q_2$  are certain analytic functions that vanish at the origin. In the new variables the equation of  $\pi^+$  will be  $\zeta = 0$ , and the equations of  $\pi^-$  will be  $\xi = 0, \eta = 0$ .

Let  $\Gamma_0$  leave the saddle-focus  $O$  into the region  $> 0$ . Then, evidently,  $\Gamma_0$  will intersect the plane  $S_1 : \zeta = d^-$  ( $0 < d^- < \varepsilon$ ) at the point  $M_1^-(0, 0, d^-)$ . Since as  $t \rightarrow +\infty$   $\Gamma_0$  tends to the saddle-focus, it will also lie in the plane  $\zeta = 0$ . It is easy to see that the plane  $S_0$ , whose equation is  $\eta = 0$ , is a secant for the trajectories of system (2) lying in the plane  $\zeta = 0$ . Consequently, as  $t \rightarrow +\infty$  the trajectory  $\Gamma_0$  intersects the plane  $S_0$  in a countable set of points  $\{M_n\}$  having the single limiting point  $O(0, 0, 0)$ .

Choose one of these points and denote it by  $M_0^+(d^+, 0, 0)$ . Clearly, one may assume that  $d^+ > 0$  and that the point  $M_0^+$  lies in the  $\varepsilon$ -neighborhood of the origin together with some neighborhood.

Let

$$\xi(t) = \xi(t, \xi_0, \zeta_0), \quad \eta(t) = \eta(t, \xi_0, \zeta_0), \quad \zeta(t) = \zeta_0 e^{\lambda t} \quad (3)$$

be a solution of system (2) passing through the point  $M_0(\xi_0, 0, \zeta_0) \in S_0$  at  $t = 0$ .

Using the known representation of solutions in a neighborhood of equilibrium states <sup>(2)</sup>, the functions  $\xi(t)$  and  $\eta(t)$  can be written in the form

$$\xi(t) = \sum_{i+j>0}^{\infty} M_{ij}^1(t) \xi_0^i \zeta_0^j e^{(i\rho+j\lambda)t},$$

(4)

$$\eta(t) = \sum_{i+j>0}^{\infty} M_{ij}^2(t) \xi_0^i \zeta_0^j e^{(i\rho+j\lambda)t},$$

where  $M_{ij}^{\alpha}(t)$  are certain trigonometric polynomials in  $\cos \omega t$  and  $\sin \omega t$ , whose coefficients in the general case are polynomials in  $t$ .

Consider the point mapping  $T_0$  of the plane  $S_0$  into the plane  $S_1$ . Evidently, it has the form

$$\begin{aligned} \xi^1 &= f_0(\xi_0, \zeta_0) = \xi \left( -\frac{1}{\lambda} \ln \frac{\zeta_0}{d^-}, \xi_0, \zeta_0 \right), \\ \eta^1 &= g_0(\xi_0, \zeta_0) = \eta \left( -\frac{1}{\lambda} \ln \frac{\zeta_0}{d^-}, \xi_0, \zeta_0 \right), \end{aligned}$$

(5)

where  $t_0 = -\frac{1}{\lambda} \ln \frac{\zeta_0}{d^-}$  is the transition time, found from the condition of intersection of the trajectory with the plane  $S_1$ , or

$$\xi_1 = \sum M_{ij}^1 \left( -\frac{1}{\lambda} \ln \frac{\zeta_0}{d^-} \right) (d^-)^j (\xi_0)^i \left( \frac{\zeta_0}{d^-} \right)^{i\nu},$$

(6)

$$\eta^1 = \sum M_{ij}^2 \left( -\frac{1}{\lambda} \ln \frac{\zeta_0}{d^-} \right) (d^-)^j (\xi_0)^i \left( \frac{\zeta_0}{d^-} \right)^{i\nu},$$

where  $\nu = -\rho\lambda^{-1}$ . Note that for  $f_0$  and  $g_0$  the following representations are valid:

$$f_0 = \xi_0 \left( \frac{\zeta_0}{d^-} \right)^{\nu} \left[ (1 + f_{01}) \cos \left( \omega_1 \ln \frac{d^-}{\zeta_0} \right) + f_{02} \sin \left( \omega_1 \ln \frac{d^-}{\zeta_0} \right) \right],$$

(7)

$$g_0 = \xi_0 \left( \frac{\zeta_0}{d^-} \right)^\nu \left[ g_{01} \cos \left( \omega_1 \ln \frac{d^-}{\zeta_0} \right) + (1 + g_{02}) \sin \left( \omega_1 \ln \frac{d^-}{\zeta_0} \right) \right],$$

where  $\omega_1 = \omega \lambda^{-1}$ , and  $f_{01}(\xi_0, \zeta_0), \dots, g_{02}(\xi_0, \zeta_0)$  are certain functions satisfying the inequality

$$|f_{01}| + |f_{02}| + |g_{01}| + |g_{02}| < \frac{1}{8}$$

for sufficiently small  $\varepsilon$ .

Denote by  $\sigma_{\delta_0}$  the domain  $\{M_0(\xi_0, 0, \zeta_0), |\xi_0 - d^+| \leq a, 0 < \zeta_0 \leq \delta_0\}$ , where  $a$  is a sufficiently small constant. Since  $\nu > 0$ , it is easy to see that for any  $\delta_1 > 0$  one can specify  $\delta_0 > 0$  such that the set  $T_0 \sigma_{\delta_0}$  will lie in the  $\delta_1$ -neighborhood of the point  $M_1^-$ .

Since  $\Gamma_0$  intersects the secant surfaces  $S_0$  and  $S_1$  without tangency, the trajectories of the system under consideration (1) passing through points of  $S_1$  close to  $M_1^-$  will intersect  $S_0$  at points close to  $M_0^+$ . We denote by  $T_1$  the mapping of  $S_0$  into  $S_1$  generated by this correspondence. From the general theorems of the theory of point mappings it follows that, whatever sufficiently small  $\sigma > 0$  may be, one can specify such a  $\delta_1 > 0$  that the images of all points from the  $\delta_1$ -neighborhood of  $M_1^-$  will lie in the  $\sigma$ -neighborhood of  $M_0^+$ . Moreover, if  $\sigma$  is sufficiently small, then the mapping  $T_1$  will be one-to-one and, in the variables  $\xi, \eta, \zeta$ , can be written in the form

$$\begin{aligned} \bar{\xi}_0 &= f_1(\xi^1, \eta^1) = d^+ + A_{11}\xi^1 + A_{12}\eta^1 + \dots, \\ \bar{\zeta}_0 &= g_1(\xi^1, \eta^1) = A_{21}\xi^1 + A_{22}\eta^1 + \dots. \end{aligned} \quad (8)$$

Let us note that, since the mapping  $T_1$  is regular,  $A = \sqrt{A_{21}^2 + A_{22}^2} \neq 0$ .

Consider the following point mapping  $T = T_1 T_0$ :

$$\begin{aligned} \bar{\xi}_0 &= f(\xi_0, \zeta_0) = f_1(f_0, g_0), \\ \bar{\zeta}_0 &= g(\xi_0, \zeta_0) = g_1(f_1, g_0). \end{aligned} \quad (9)$$

Obviously, for sufficiently small  $\delta_0$  the mapping is defined in  $\sigma_{\delta_0}$ , and  $\sigma_{\delta_0}$  is mapped into the  $\sigma$ -neighborhood of the point  $M_0^+$ .

We shall seek the fixed points of the mapping  $T$ . First make the substitution

$$\omega_1 \ln \frac{d^-}{\zeta_0} = 2\pi n + \varphi,$$

where  $n = E \left( \frac{\omega_1}{2\pi} \ln \frac{d^-}{\zeta_0} \right)$ ,  $0 \leq \varphi < 2\pi$ . By this substitution the domain  $\sigma_{\delta_0}$  is split into a countable set of layers  $K_n : \{|\xi_0 - d^+| \leq a, d^- e^{-2\pi(n+1)/\omega_1} \leq \zeta_0 <$

$d^- e^{-2\pi n/\omega_1}$ }. We shall prove that in each layer  $K_n$ , where  $n$  is greater than a certain sufficiently large  $N$ , the mapping  $T$  has two fixed points.

In the new variables the equations for finding the coordinates of the fixed points of  $K_n$  will be written in the form

$$\xi_0 = f(\xi_0, d^- e^{-(2\pi n + \varphi)/\omega_1}), \quad d^- e^{-(2\pi n + \varphi)/\omega_1} = g(\xi_0, d^- e^{-(2\pi n + \varphi)/\omega_1}). \quad (10)$$

Taking into account the expressions for  $f_0$  and  $g_0$ , equations (10) can also be written in the form

$$\xi_0 = d^+ + e^{-2\pi n\nu/\omega_1} p(\xi_0, \varphi, n),$$

$$d^- e^{-(2\pi n + \varphi)(1-\nu)/\omega_1} = \xi_0 [\bar{A}_{21}(1 + q_1(\xi_0, \varphi, n)) \cos \varphi + \bar{A}_{22}(1 + q_2(\xi_0, \varphi, n)) \sin \varphi], \quad (11)$$

where the functions  $p, q_1$ , and  $q_2$  are analytic in  $K_n$ ;  $q_1$  and  $q_2$  tend to zero as  $n \rightarrow \infty$  together with their derivatives;  $\bar{A} = \sqrt{\bar{A}_{21}^2 + \bar{A}_{22}^2} \neq 0$ . It is not difficult to see that the first equation of system (11) is solvable with respect to  $\xi_0$ :

$$\xi_0 = d^+ + e^{-2\pi n\nu/\omega_1} \psi_n(\varphi), \quad (12)$$

where  $\psi_n(\varphi)$  is an analytic function in the interval of variation of  $\varphi$  under consideration and has bounded derivatives there. Substituting the expression for  $\xi_0$  into the second equation, we obtain an equation for finding

of the second coordinate of the fixed point. This equation can be written in the form

$$\frac{d^-}{d^+} e^{-(2\pi n + \varphi)(1-\nu)/\omega_1} = \bar{A} (1 + \alpha_1(n, \varphi)) \sin(\varphi + \theta_0 + d_2(n, \varphi)), \quad (13)$$

where  $\alpha_1$  and  $\alpha_2 \rightarrow 0$  as  $n \rightarrow \infty$ , together with their derivatives, and  $\theta_0 = \arctg \bar{A}_{21}/\bar{A}_{22}$ .

Since, by the hypothesis of the theorem,  $\nu < 1$ , it is easy to see that, for any sufficiently large  $n$ , equation (13) will have two positive solutions  $\varphi_n^1$  and  $\varphi_n^2$ , satisfying the conditions

$$\lim_{n \rightarrow \infty} \varphi_n^1 = -\theta_0 \text{ (or } 2\pi - \theta_0),$$

$$\lim_{n \rightarrow \infty} \varphi_n^2 = \pi - \theta_0 \text{ (or } \pm 3\pi - \theta_0). \quad (14)$$

Using the well-known relation between periodic motions and fixed points of the corresponding transformation, we obtain that in any neighborhood of  $\Gamma_0$  there is a countable set of periodic solutions.

Let us show that each fixed point  $M_{ni}^*(\xi_n^i, 0, \varphi_n^i)$  of the mapping  $T$  ( $i$  either 1 or 2), sufficiently close to  $M_0^+$ , is a fixed point of saddle type.

As is not difficult to see, the linearized point mapping  $T$  in a neighborhood of the fixed point  $M_{ni}^*$ , in the variables  $\xi_0$  and  $\varphi$ , will be written in the form

$$d\bar{\xi} = C_{11}(n, i) d\xi + C_{12}(n, i) d\varphi,$$

$$d^- e^{-(Q\pi n + \varphi_n^i)(1-\nu)/\omega_1} d\bar{\varphi} = C_{21}(n, i) d\xi + C_{22}(n, i) d\varphi, \quad (15)$$

where  $|C_{11}(n, i)| + |C_{12}(n, i)| \rightarrow 0$  as  $n \rightarrow \infty$ ,  $C_{21}$  is bounded for all  $n > N$ , and for  $C_{22}$  the representation

$$C_{22}(n, i) = (-1)^{i+1} Ad^{+\nu} + \beta(n), \quad (16)$$

holds, where  $\beta(n) \rightarrow 0$  as  $n \rightarrow \infty$ . From the representation of the characteristic equation in the form

$$d^- e^{-(2\pi n + \varphi_n^i)(1-\nu)/\omega_1} z^2 + [(-1)^i Ad^{+\nu} + \beta_1(n)] z + \beta_2(n) = 0, \quad (17)$$

where  $\beta_1(n)$  and  $\beta_2(n) \rightarrow 0$  as  $n \rightarrow \infty$ , it follows that, for sufficiently large  $n$ , equation (17) will have two real roots, one of which is greater than unity in absolute value and the other less than unity.

Thus it has been proved that the periodic motions found, sufficiently close to  $\Gamma_0$ , will be unstable solutions of saddle type.

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*Note: Figure translations are in progress. See original paper for figures.*

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