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Abstract

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METRIC PROPERTIES OF MAPPINGS OF THE CLASS $BL^{3/2}$

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1°. In the present note we set forth results obtained for homeomorphic spatial mappings on the basis of a further development of the method of papers (^{1,2}). In those papers, in spatial domains, a relative distance was introduced, an upper estimate was obtained for the distortion of this distance in a closed domain for homeomorphic mappings of the class $BL^{3/2}$, and on the basis of this estimate a number of properties of mappings were established; in particular, certain conclusions were drawn concerning the correspondence of boundaries under mappings.

Here we begin with an example of a spatial homeomorphic mapping showing the impossibility, under the conditions of papers (^{1,2}), of obtaining a nontrivial two-sided estimate of the distortion of relative distances for homeomorphic mappings of the class under consideration. Meanwhile, the most complete conclusions of a metric character about the classes of mappings under consideration are possible only when two-sided estimates are available. This justifies the introduction here of another definition of relative distance. For the new distance, a two-sided estimate of its distortion already becomes possible (see Theorem 1). Our distance is metric; locally (inside the domain) it coincides with the Euclidean one, but, unlike the distance in (^{1,2}), the completion of a domain with respect to the new distance adjoins the ordinary boundary of the domain as the single "boundary element." The conclusion about boundary correspondence under mappings, which follows from this estimate, is, of course, of a completely trivial character (the boundary goes into the boundary), but, since the estimate turns out to be valid up to the boundary, it makes it possible to estimate the rate of approach to the boundary of the image point as a function of the rate of approach to the boundary of the preimage point. This makes it possible to establish a number of essential metric properties of the classes of mappings under consideration, analogous to the properties of plane mappings of the classes BL and BL_k. In doing this it proved possible to follow, in the main, the method developed by G. D. Suvorov ($^{3-8}$) for obtaining such properties in the plane case.

We also note that throughout the whole paper we follow the general method for studying similar questions set forth in the note (^{1}).

2°. We shall denote by $y = f(x)$, where $y = (y_1, y_2, y_3)$, $x = (x_1, x_2, x_3)$, vector functions defined in domains of three-dimensional Euclidean space E^3. Let ∂D

be the boundary of the domain D; $\rho(x', x'')$ the distance between the points x' and x'' in E^3; $\overline{M}$ the closure of the set M in E^3; $\{x : \pi(x)\}$ the set of points x for which the condition $\pi(x)$ is satisfied.

3°. Example. Suppose that in E^3, in the half-plane $x_1 \geq 0$ of the plane x_1Ox_2, there is a simply connected domain A, homeomorphically mapped by the functions

$$y_1 = f_1(x_1, x_2), \qquad y_2 = f_2(x_1, x_2)$$

onto a domain B, lying in the half-plane $y_1 \geq 0$ of the plane y_1Oy_2. By rotating the domains A and B about the axes x_2 and y_2, respectively, we can obtain a spatial homeomorphic mapping of some domain D onto a domain Δ, given by a vector func-

by $y = f(x)$, written in coordinate form as follows:

$$y_1 = \frac{x_1}{u} f_1(u, x_2), \qquad y_2 = f_2(u, x_2), \qquad y_3 = \frac{x_3}{u} f_1(u, x_2), \tag{1}$$

where $u = \sqrt{x_1^2 + x_3^2}$. A calculation shows that the Dirichlet integral (5) of this mapping is equal to

$$I(f, D) = 2\pi \int_A \left[\sum_{i,j=1}^{2} \left(\frac{\partial f_i(x_1, x_2)}{\partial x_j} \right)^2 + \frac{f_1^2(x_1, x_2)}{x_1^2} \right]^{3/2} x_1 \, dx_1 dx_2. \tag{2}$$

We now choose the domain A and the functions f_1 and f_2 in the following way: obtain the domain A from the square with vertices at the points $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$, by discarding that part of it which is cut off by the curve

$$x_1 = \varphi(x_2) \equiv \sqrt{x_2 - 1} \tag{3}$$

and contains the point $(0, 2)$. Consider the curve given by the equation

$$x_1 = \psi(x_2) \equiv (2 - x_2)^{\alpha}, \qquad {}^2\!/_3 < \alpha < 1. \tag{4}$$

The curves (3) and (4) intersect at $x_2 = a$, $1 < a < 2$. Deform the closed domain $\bar{A}$ so that its points for which $0 \leq x_2 \leq a$ remain fixed, while for $a \leq x_2 \leq 2$ the points of the curve (3) pass into the points of the curve (4), the deformation being carried out linearly along straight lines parallel to the x_1-axis. As a result we obtain the mapping

$$y_1 = f_1(x_1, x_2) \equiv \begin{cases} 2 - (2 - x_1)\dfrac{2 - \psi(x_2)}{2 - \varphi(x_2)}, & a \leq x_2 \leq 2, \\ x_1, & 0 \leq x_2 \leq a, \end{cases}$$

$$y_2 = f_2(x_1, x_2) \equiv x_2.$$

Using (2), it is easy to verify that the mapping (1) and its inverse have bounded Dirichlet integral. At the same time, the mapping inverse to (1) carries the point $y = (0, 2, 0)$ into the circle of unit radius in the plane $x_2 = 2$. For the domains D and Δ, the boundary points can be identified with boundary elements with respect to the relative distance of the works (^{1}, ^{2}). It is easy to see that if the lower estimate of distortion of relative distance under the mapping (1) could be carried out in the closed domain $\bar{D}$, then the inverse mapping to (1) could be extended in the closed domain $\bar{\Delta}$ to a continuous one, which is impossible*.

4°. **Definition 1.** By the **relative distance** between points x', x'' of a domain $D \supset O$ we mean the quantity

$$\sigma(x', x''; D) = \min \left[\rho(x', x''), \inf_{x \in F} \sup \rho(x', \partial D) \right],$$

where the infimum is taken over all possible closed (relative to D) sets F separating** the points x', x'' from O in D and such that, if $\bar{F} \cap \partial D$ is empty, then F separates x', x'' also from ∂D in $\bar{D}$.

Here and below we assume that the set ∂D is nonempty.

The relative distance introduced in the domain D is a metric distance. Denote by $\tilde{D}$ the completion of the domain D with respect to the metric σ. Then $\tilde{D}$ contains D, and $\tilde{D} \setminus D = \partial \tilde{D}$ is a single point in the metric σ; this point may be identified with the ordinary boundary of the domain D (if a sequence of points of D has all its accumulation points in the metric ρ on the boundary of D, then it converges in the metric σ to the point $\partial \tilde{D}$).

* The mapping (1) does not belong to the class T_μ considered in the work of V. A. Zorich (^{9}).

** Let D be an open or closed domain. We say that a set F **separates** a set M_1 from a set M_2 in D, if $F \subset D$ and every continuous curve lying in D and containing points from M_1 and M_2 also contains points from F.

We introduce for consideration the following sets, assuming that $O \in D$:

$$G_\alpha(D) = \{x : \ \rho(x, \partial D) > \alpha\} \cap D, \qquad F_\alpha(D) = \{x : \ \rho(x, \partial D) \geq \alpha\} \cap D$$

for $0 < \alpha < \rho(0, \partial D)$; $G^0_\alpha(D)$, $F^0_\alpha(D)$ are the components of the sets $G_\alpha(D)$ and $F_\alpha(D)$ containing the point O; $S^0_\alpha(D) = F^0_\alpha(D) \setminus G_\alpha(D)$; analogously (with ρ replaced by σ and ∂D by $\partial \widetilde{D}$) we introduce the sets $G_\alpha(D, \sigma)$, $F_\alpha(D, \sigma)$, $G^0_\alpha(D, \sigma)$, $F^0_\alpha(D, \sigma)$, $S^0_\alpha(D, \sigma)$. In the notation of these sets D is sometimes omitted.

Lemma 1. Let the domain D be bounded and $O \in D$. Then

a) $G_\alpha(\sigma) = G^0_\alpha(\sigma) = G^0_\alpha$;

b) $S^0_\alpha(\sigma) = F_\alpha(\sigma) \setminus G_\alpha(\sigma) = F^0_\alpha \setminus G^0_\alpha$;

c) $F_\alpha(\sigma) = F^0_\alpha(\sigma) = F^0_\alpha$;

d) $F^0_\alpha = \bigcap\limits_{0<\varepsilon<\varepsilon_1} \overline{G}^0_{\alpha-\varepsilon}$ $(\alpha - \varepsilon_1 > 0)$.

Corollary. Let the domain D be bounded and $O \in D$. Then: a) if $x \in S^0_\alpha$, then $\sigma(x, \partial\widetilde{D}; \widetilde{D}) = \rho(x, \partial D)$; b) if $x', x'' \in F^0_\alpha$ and $\rho(x', x'') \leq \alpha$, then $\sigma(x', x''; D) = \rho(x', x'')$.

5°. **Definition 2.** A continuous vector function $y = f(x)$, given in a domain $D \subset E^3$, belongs to the class of functions $BL^{3/2}_K$ if it has a bounded Dirichlet integral

$$I(f, D) = \int_D \left[\sum_{i,j=1}^{3} \left(\frac{\partial f_i}{\partial x_i} \right)^2 \right]^{3/2} dx_1 dx_2 dx_3 \leq K < \infty, \tag{5}$$

where the derivatives are understood in the sense of S. L. Sobolev.

If $f \in BL^{3/2}_K$ for some K, then $f \in BL^{3/2}$. If the function $y = f(x) \in BL^{3/2}_K$ maps homeomorphically the domain D, containing O, onto a domain Δ; $f^{-1}(y) \in BL^{3/2}_K$ in Δ; $f(0) = 0$, then $f(x) \in (BL)^{3/2}_K$.

Definition 3. If the domain D is bounded, contains a ball of radius $\delta > 0$ with center at the origin of coordinates, and if the diameter of any component of the boundary of the domain D is not less than $\beta > 0$, then we say that $D \in A(\beta, \delta)$.

Let

$$\exp[-4M_0 K \sigma^{-3}] \equiv \varphi_1(\sigma), \qquad [4M_0 K]^{-1/3} \ln^{-1/3} \frac{1}{\sigma} \equiv \varphi_2(\sigma),$$

where

$$M_0 = 2^6 \pi^2 \Gamma(3/4)^{-4},$$

Γ is Euler’ s gamma function, $K = \text{const}$.

Theorem 1. Let $y = f(x) \in (BL)^{3/2}_K$ and map the domain $D \in A(\beta_0, \delta_0)$ onto the domain $\Delta \in A(\beta_1, \delta_1)$. Then $f(x)$ can be extended by continuity to a homeomorphic mapping $\widetilde{\Delta} = f(\widetilde{D})$, and for any points $x', x'' \in \widetilde{D}$ for which

$$\sigma(x', x''; \widetilde{D}) < \min[a_1, \varphi_1(a_2)],$$

where

$$a_1 = \min\left[\frac{1}{5}\delta_0^2, \frac{1}{4}\beta_0^2, \gamma\right], \qquad a_2 = \min\left[\frac{1}{5}\delta_1^2, \frac{1}{4}\beta_1^2, \gamma\right], \qquad \gamma = 2(3-2\sqrt{2}),$$

the two-sided estimate of the distortion of the relative distance holds:

$$\varphi_1(\sigma(x', x''; \widetilde{D})) \leq \sigma(f(x'), f(x''); \widetilde{\Delta}) \leq \varphi_2(\sigma(x', x''; \widetilde{D})).$$

6°. The following theorems 2–4 can be obtained from Theorem 1 without invoking any additional metric properties of the mappings under consideration.

Theorem 2. Let $y = f(x) \in (BL)_K^{3/2}$ and map the domain $D \in A(\beta_0, \delta_0)$ onto the domain $\Delta \in A(\beta_1, \delta_1)$.

If $\alpha < \min(\alpha_1, \varphi_1(\alpha_2), \delta_0]$, then

$$F^0_{\varphi_1(\alpha)}(\Delta) \supset f(F^0_\alpha(D)) \supset F^0_{\varphi_2(\alpha)}(\Delta).$$

Theorem 3. Let $\varphi = f(x) \in (BL)_K^{3/2}$ and map the domain $D \in A(\beta_0, \delta_0)$ onto the domain $\Delta \in A(\beta_1, \delta_1)$. For any points $x', x'' \in F^0_\alpha(D)$ $(\alpha < \rho(0, \partial D))$, for which

$$\rho(x', x'') < \min\{a_1, \varphi_1(a_2), \alpha, \varphi_1[\varphi_1(\alpha)]\},$$

there is a two-sided estimate for the distortion of distance inside the domain:

$$\varphi_1(\rho(x', x'')) \leq \rho(f(x'), f(x'')) \leq \varphi_2(\rho(x', x'')).$$

Corollary. The family of mappings $\{f\}$ of domains $\{D_f\}$ onto domains $\{\Delta_f\}$, $f \in (BL)_K^{3/2}$, $D_f \in A(\beta_0, \delta_0)$, $\Delta_f \in A(\beta_1, \delta_1)$, has the property of equicontinuity and uniform openness with respect to the origin O inside the domains $\{D_f\}$*.

The estimate given above also makes it possible to establish the orders of equicontinuity and uniform openness of the family $\{f\}$. This corollary, taking into account the result in (8), makes it possible to obtain the following analogue of the Carathéodory theorem (on conformal mappings of domains with variable boundaries).

Theorem 4. Let $\{f_n\}$ $(n = 1, 2, ...)$ be a sequence of functions, each of which maps a domain D_n onto a domain Δ_n, where $f_n \in (BL)_K^{3/2}$, $D_n \in A(\beta_0, \delta_0)$, $\Delta_n \in A(\beta_1, \delta_1)$ for every n, and the sequences of domains $\{D_n\}$ and $\{\Delta_n\}$ are uniformly bounded. Then one can extract from $\{f_n\}$ a sequence $\{f_{n_p}\}$ $(p = 1, 2, ...)$ such that:

1. $\{D_{n_p}\}$ $(\{\Delta_{n_p}\})$ converges to the kernel D_0 (Δ_0) with respect to the point O.
2. The sequence $\{f_{n_p}\}$ $(\{f_{n_p}^{-1}\})$ converges uniformly inside D_0 (Δ_0) to a mapping f (f^{-1}) of class $(BL)_K^{3/2}$ of the domain D_0 (Δ_0) onto Δ_0 (D_0).

Remark 1. If all domains of the sequence $\{D_n\}$ coincide, then they coincide with the kernel D_0. In this case Theorem 4 establishes the compactness property in itself of the class $(BL)_K^{3/2}$ under the assumptions of the theorem.

Remark 2. Theorem 4 can be generalized in various ways to a broad class of mappings of unbounded domains. For example, using inversion with respect to a sphere, which is a conformal mapping, one can obtain the following. Suppose that each domain $D_n \in \{D_n\}$ has a connected boundary, contains a ball of radius δ_0 with center at the point O, and the exterior of D_n contains a ball of radius δ_0 with center at the point $x^{(n)}$, where the set $\{x^{(n)}\}$ is bounded; the domains from $\{\Delta_n\}$ have an analogous property; f_n is a Q-quasiconformal mapping for every n, $f_n(0) = 0$, and Q does not depend on n. Then Theorem 4 is valid under these assumptions if the assertion that $f, f^{-1} \in BL_K^{3/2}$ is replaced by the assertion that f is a Q-quasiconformal mapping.

Theorem 5. Let $y = f(x) \in BL^{3/2}$ and let it map homeomorphically a ball with the center removed onto some domain. Then the function $y = f(x)$ can be continuously extended at the center of the ball.

Remark 3. The results presented here can also be generalized to mappings in n-dimensional Euclidean space.

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* We formulate this property analogously to how this was done in (4).

Note: Figure translations are in progress. See original paper for figures.

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