Soviet-era science, translated into English

Reports of the Academy of Sciences of the USSR

1965

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Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1965. Volume 163, No. 4

MATHEMATICS

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ON A PERTURBATION METHOD IN THE THEORY OF DYNAMIC STABILITY OF SYSTEMS WITH DISTRIBUTED PARAMETERS

(Presented by Academician V. I. Smirnov on 18 I 1965)

1. Many problems of dynamic stability of systems with distributed parameters (1) lead to the investigation, in a separable Hilbert space H, of the linear Hamiltonian equation

$$i\frac{d}{dt}Fx = [I + \mathcal{H}(\tau)]x, \qquad \tau = \theta t, \qquad i = \sqrt{-1}, \tag{1}$$

whose coefficients satisfy the conditions:

a) F is a self-adjoint, completely continuous operator in H, having an (unbounded) inverse F^{-1};

b) the 2π-periodic symmetric operator function $\mathcal{H}(\tau)$ is subordinate to the operator F^{-1} in the sense that the operator function $F^{-1}\mathcal{H}(\tau)$ is uniformly continuous on the interval $[0, 2\pi]$ and has a uniformly continuous derivative $\frac{d}{d\tau}F^{-1}\mathcal{H}(\tau)$.

The numerical parameter θ characterizes the frequency of the perturbation, and I is the identity operator.

By a **solution** of equation (1) is meant any continuous function $x(t)$ for which the function $Fx(t)$ has, for almost all $t \in [0, 2\pi/\theta]$, a strong derivative and, for these values of t, satisfies equation (1). It is not hard to see that for every $x_0 \in H$ there exists a unique solution $x(t)$ of equation (1) determined by the condition $x(0) = x_0$.

Theorem 1. *The operator solution $X(t)$ of the Hamiltonian equation (1), defined from the solution $x(t)$ by the formula $x(t) = X(t)x(0)$, is representable in the form*

$$X(t) = U(t) + V(t),$$

where for every t the operator $U(t)$ is unitary, and $V(t)$ is completely continuous.

If the operator $[I + \mathcal{H}(\tau)]$ is positive definite for all $\tau \in [0, 2\pi]$ and $S(t)$ is the positive root of the operator $[I + \mathcal{H}(\theta t)]$, then the proof of the theorem is based on the fact that the operator function $S(t)X(t)S^{-1}(t)$ satisfies an equation differing only slightly from the Schrödinger equation. Using this circumstance, it is not difficult to prove the theorem in the general case.

It follows from Theorem 1 (^{2}) that the spectrum of the monodromy operator $X(2\pi\theta^{-1})$ of equation (1), lying outside the unit circle, can consist only of isolated eigenvalues of finite multiplicity.

2. The problem of computing the isolated eigenvalues of the operator $X(2\pi\theta^{-1})$ can be reformulated as the problem of finding, by the perturbation method, the eigenvalues of a certain self-adjoint operator under perturbation by a bounded operator.

Denote by D the linear set of 2π-periodic functions, continuously differentiable on the interval $[0, 2\pi]$, such that the values of these functions and of their derivatives belong to the domain of definition of the operator F^{-1} and are transformed by this operator into continuous functions. On the set D define the scalar product by the formula

$$[x, y] = (2\pi)^{-1} \int_0^{\pi} (F^{-1}x(\tau), F^{-1}y(\tau))\, d\tau.$$

The completion of the set D in the metric generated by this scalar product will be a separable Hilbert space, which we denote by L_2. It is not difficult to see that the operator $\widetilde{S}_0$, generated on the set D by the differential expression

$$\widetilde{S}_0 = -i\frac{d}{d\tau} - \frac{1}{\theta}F^{-1},$$

is a symmetric operator, and its closure S_0 is a self-adjoint operator in L_2. In view of condition b), the operator generated by the relation

$$S_1 y = -\frac{1}{\theta}(F^{-1}\mathcal{H}(\tau))^* y(\tau)$$

is defined everywhere in L_2 and is bounded.

Theorem 2. *If λ is an eigenvalue of the operator $S_0 + S_1$, then the number $\exp(2\pi i\lambda)$ is an eigenvalue of the operator $X(2\pi\theta^{-1})$. Conversely, if ρ is an eigenvalue of the operator $X(2\pi\theta^{-1})$, then the numbers $(2\pi i)^{-1}\ln\rho$, for any branch of the logarithm, are eigenvalues of the operator $S_0 + S_1$. If the point $\exp(2\pi i\lambda)$ belongs to the regularity domain of the operator $X(2\pi\theta^{-1})$, then the points $\lambda + k$, k an integer, belong to the regularity domain of the operator $S_0 + S_1$.*

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In what follows the numbers $\alpha = i\lambda$, where λ are eigenvalues of the operator $S_0 + S_1$, will be called the **characteristic exponents** of equation (1).

3. Consider the problem of approximately computing isolated eigenvalues of the operator $S_0 + S_1$. Let λ_0 be an arbitrary fixed point of the complex plane and let δ be some positive number. Denote by Ω the set of points λ of the real axis satisfying the condition $|\lambda - \lambda_0| > \delta$. Let E_λ be the spectral family of the operator S_0, and

$$P_\delta = \int_\Omega dE_\lambda.$$

Obviously, in the subspace $P_\delta L_2$ the operator $S_0 - \lambda_0 I$ (where I is the identity operator in L_2) has a bounded inverse. We denote it by $(S_0 - \lambda_0 I)^{-1} P_\delta$.

Theorem 3. *In order that the number λ, satisfying the condition*

$$\delta^{-1} \|(\lambda - \lambda_0) I - S_1\| < 1,$$

be an eigenvalue of the operator $S_0 + S_1$, it is necessary and sufficient that this number be an eigenvalue of the operator $\Phi(\delta, \lambda, \lambda_0)$, generated in the subspace $(I - P_\delta) L_2$ by the matrix with matrix elements

$$\Phi_{kl}(\delta, \lambda, \lambda_0) = [(S_0 + S_1) x_l, x_k] -$$

[
[S_1(I+(S_0- 0 I)^{-1}P{ }{S_1-(- _0)I})^{-1}(S_0- 0 I)^{-1}P{ } S_1x_l,x_k],]

where $\{x_k\}$, $k = 1, 2, \ldots$, is an orthonormal basis in the subspace $(I - P_\delta) L_2$.

We outline the proof. Let λ be an eigenvalue of the operator $S_0 + S_1$ and x^0 the corresponding eigenvector. From the identity

$$(S_0 - \lambda_0 I) x^0 = (\lambda - \lambda_0) x^0 - S_1 x^0 \tag{2}$$

the element $P_\delta x^0$ is uniquely determined:

$$P_\delta x^0 = M x^0,$$

where the operator M has the form

$$M = (S_0 - \lambda_0 I)^{-1} P_\delta \{(\lambda - \lambda_0) I - S_1\}.$$

Hence we find the element x^0 through its projection $(I-P_\delta)x^0$ in the subspace $(I-P_\delta)L_2$:

$$x^0 = (I-M)^{-1}(I-P_\delta)x^0. \tag{3}$$

Applying the operator $S_0-\lambda_0 I$ to both sides of identity (2) and using relation (3), we easily arrive at the necessity in Theorem 3. Sufficiency is obtained by carrying out the arguments in the reverse order.

In view of the assumptions made above, it is not difficult to obtain the estimate

$$\|\Phi(\delta,\lambda,\lambda_0)-(I-P_\delta)(S_0+S_1)(I-P_\delta)\| \le \delta^{-1}\|S_1\|^2\left(1-\delta^{-1}\|(\lambda-\lambda_0)I-S_1\|\right)^{-1}. \tag{4}$$

If the right-hand side in inequality (4) is small, then the operator $(I-P_\delta)(S_0+S_1)(I-P_\delta)$ is the "principal" part of the operator $\Phi(\delta,\lambda,\lambda_0)$. Such a situation can be achieved in two ways: either by the smallness of the quantities $\|S_1\|$ and $|\lambda-\lambda_0|$ for fixed δ, or by choosing δ sufficiently large for given estimates of the quantities $\|S_1\|$ and $|\lambda-\lambda_0|$. In the latter case these quantities need not be "small." However, increasing the number δ complicates the study of the operator $(I-P_\delta)(S_0+S_1)(I-P_\delta)$.

4. Let us give some consequences of applying Theorem 3 to equation (1) in the case of finite perturbations*.

Consider the system

$$i\lambda_k^{-1}\frac{df_k}{dt} = f_k + \sum_{l=-N}^{N} h_{kl}(\theta t)f_l, \qquad k=\pm1,\pm2,\dots,\pm N, \tag{5}$$

obtained from equation (1) by applying the Galerkin method. Here $h_{kl}(\tau) = (\mathcal{H}(\tau)a_l, a_k)$, λ_k are the eigenvalues of the operator F^{-1}, and a_k are the corresponding eigen-elements. Denote by $\rho(\mathcal{H},\theta)$ and $\rho_N(\mathcal{H},\theta)$ the largest real parts of the characteristic exponents of equation (1) and system (5), respectively. Suppose that on both the negative and the positive semiaxis there exists a sequence of arcs (intervals of regular points of the operator F^{-1}) whose lengths increase without bound.

Theorem 4. For any positive numbers ε, C, θ_0, and θ_1, where $\theta_0<\theta_1$, there exists a natural number N_0 such that, for all $N>N_0$, the estimate

$$|\rho(\mathcal{H},\theta)-\rho_N(\mathcal{H},\theta)|<\varepsilon$$

holds for all $\mathcal{H}$ and θ satisfying the conditions

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$$\max_{\tau\in[0,2\pi]} \|F^{-1}\mathcal{H}(\tau)\| \leq C, \qquad \theta_0 \leq \theta \leq \theta_1.$$

The proof is based on the fact that the function $\mathcal{H}(\tau)$ can be approximated by a finite segment of its Fourier series. For the approximating function one can choose δ so large that the largest real part of the eigenvalues of the operator $(I - P_\delta)(S_0 + S_1)(I - P_\delta)$ will be arbitrarily close to $\rho(\mathcal{H}, \theta)$. In view of the assumptions made above, for any finite δ this operator decomposes into an orthogonal sum of finite-dimensional operators, from which the assertion of the theorem is easily derived.

Under the assumptions made, one can prove that the solution of the "truncated" system (5) corresponding to the characteristic exponent with the largest real part converges, as $N \to \infty$, on each finite interval $[0, T]$ to the corresponding solution of equation (1). This assertion is, in essence, a justification of the Galerkin method for systems of the type considered.

* The results of this section were obtained independently and by another method by V. I. Derguzov.

Suppose that the operator function $\mathcal{H}(\tau)$, in addition to τ, is a continuous function of certain parameters. The simplest examples of the systems considered show that the set of parameters corresponding to solutions of equation (1) that grow unboundedly as $t \to \infty$ is often dense in the parameter space of this equation. This circumstance is connected with the fact that the set of characteristic exponents of equation (1) does not, generally speaking, change continuously under a continuous change of the coefficients of the equation. From Theorem 4, however, it follows that the function $\rho(\mathcal{H}, \theta)$, which coincides with the largest real part of the characteristic exponents, is a continuous function of its arguments. Hence, in particular, it follows that for every $a \geq 0$ the set in the parameter space of equation (1) defined by the inequality $\rho(\mathcal{H}, \theta) > a$ is open. We note that this set coincides with the set of parameters for which the general solution $x(t)$ of equation (1), as $t \to \infty$, grows no more slowly than $\exp(a\theta^{-1}t)$.

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Received
4 I 1965

CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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