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Abstract

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THE INVARIANT MEANING OF THE RIEMANN INVARIANTS

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In a number of works, summarized in ([1]), L. V. Ovsyannikov applied the theory of Lie groups to the study of the group properties of differential equations. Cartan' s method of exterior forms ([2]) can also be applied to this same problem. The idea of applying this method to the study of the group properties of differential equations is as follows.

As is known ([3]), the structural equations of an infinite group have the form (D is the symbol of exterior differentiation):

$$D\theta^i = c^i_{jk}[\theta^j\theta^k] + a^i_{j\lambda}[\theta^j\theta^\lambda], \tag{1*}$$

where θ^i are the basic invariant forms of the group; [ ] denotes exterior multiplication; θ^λ are certain additional forms; $c^i_{jk}, a^i_{j\lambda}$ are functions of the invariants or, in particular, constants. If the group is finite, then $a^i_{j\lambda} = 0$.

On the other hand, suppose we have a system of equations, for example

$$u^k_t + a^k_l(x, t, u^1, \dots, u^{n-2})\, u^l_x = 0 \quad (l, k = 1, 2, \dots, n-2), \tag{2*}$$

where $u^1, u^2, \dots, u^{n-2}$ are functions; x, t are independent variables. Let us write the obvious relations

$$du^k - u^k_t\, dt - u^k_x\, dx = du^k + a^k_l u^l_x\, dt - u^k_x\, dx = 0.$$

Denote these Pfaffian forms by ω^k and complete them to a full basis of the space $T(x, t, u^1, \dots, u^{n-2})$ by the differentials dx, dt or by their linear combinations; we obtain forms ω^i $(i = 1, 2, \dots, n)$. Differentiating the forms ω^i exteriorly, we obtain a system of exterior differential equations

$$D\omega^i = b^i_{jk}[\omega^j\omega^k] + d^i_{j\lambda}[\omega^j\omega^\lambda], \tag{3*}$$

where $b^i_{jk}, d^i_{j\lambda}$ depend on $(x, t, u^1, \ldots, u^{n-2})$ and on the parametric derivatives u^l_x; ω^λ are Pfaffian forms depending on the differentials of the parametric derivatives du^l_x.

Introducing new forms $\Omega^i = A^i_j \omega^j$, we can, by means of the elements of the matrix $A(A^i_j)$, carry out a canonization of the coefficients b^i_{jk} ([2]). If in this process the system (3*) takes the form

$$D\Omega^i = B^i_{jk}[\Omega^j \Omega^k] \quad (B^i_{jk} = \text{const or invariants}), \tag{4*}$$

then we obtain that the Ω^i are basic invariant forms of some finite group G, attached to the original system (2*). As an example, let us consider the equations describing one-dimensional flows of a gas with plane waves.

One-dimensional gas flows with plane waves are described by the system of equations (the notation is standard):

$$u_t + uu_x + p_x/\rho = 0; \tag{1}$$

$$p_t + up_x + \gamma p u_x = 0; \tag{2}$$

$$\rho_t + u\rho_x + \rho u_x = 0. \tag{3}$$

For this system of equations, under the assumption of isentropy, the concept of Riemann invariants is well known (see, for example, ([4])):

$$I^+ = u + \int \frac{dp}{\rho a}, \qquad I^- = u - \int \frac{dp}{\rho a}, \tag{4}$$

where a is the speed of sound.

These variables were introduced by Riemann on the basis of physical considerations. We shall show that, in fact, the Riemann invariants have an explicitly expressed group character, and it is precisely this fact that justifies the use of the very word "invariant." To investigate the group properties of the system (3), we apply Cartan' s method of exterior forms.

Let us write the system (1)–(3) in the form of Pfaffian forms:

$$du - u_x\,dx + \left(uu_x + \frac{p_x}{\rho}\right) dt = \tilde{\omega}^6; \tag{5}$$

$$dp - p_x\,dx + (up_x + \gamma p u_x)\,dt = \tilde{\omega}^7; \tag{6}$$

$$d\rho - \rho_x\, dx + (u\rho_x + \rho u_x)\, dt = \tilde{\omega}^8. \tag{7}$$

Add here two more Pfaffian forms associated with the characteristics:

$$dx - (u - a)dt = \omega^1; \tag{8}$$

$$dx - (u + a)dt = \omega^2. \tag{9}$$

The characteristic system for the forms (5)–(9) contains 8 variables (forms), corresponding to $x, t, u, p, \rho, u_x, p_x, \rho_x$, and, consequently, exterior differentiation of the system (5)–(9) introduces another three forms, linear with respect to $du_x, dp_x, d\rho_x$. To simplify the system of exterior differential equations, introduce the forms

$$\omega^6 = \frac{1}{4\gamma p}\left(\rho a\tilde{\omega}^6 - \tilde{\omega}^7\right); \tag{10}$$

$$\omega^7 = \frac{1}{4\gamma p}\left(\rho a\tilde{\omega}^6 + \tilde{\omega}^7\right); \tag{11}$$

$$\omega^8 = \frac{1}{4\gamma\rho p}\left(\rho\tilde{\omega}^7 - \gamma p\tilde{\omega}^8\right). \tag{12}$$

The system of exterior differential equations for the forms $\omega^1, \omega^2, \omega^6, \omega^7, \omega^8$ has the form

$$\begin{aligned}
D\omega^1 &= [\omega\omega^1] + a^1_{16}[\omega^1\omega^6] + a^1_{17}[\omega^1\omega^7] + a^1_{26}[\omega^2\omega^6] + a^1_{27}[\omega^2\omega^7] + a^1_{28}[\omega^2\omega^8];\\
D\omega^2 &= [\omega\omega^2] - a^1_{16}[\omega^2\omega^6] - a^1_{16}[\omega^2\omega^7] - a^1_{27}[\omega^1\omega^6] - a^1_{26}[\omega^1\omega^7] + a^1_{28}[\omega^1\omega^8];\\
D\omega^6 &= [\omega^3\omega^1] + [\theta^6_6\omega^6] + b^6_{27}[\omega^2\omega^7] + b^6_{28}[\omega^2\omega^8] + b^6_{78}[\omega^7\omega^8];\\
D\omega^7 &= [\omega^4\omega^2] + [\theta^7_7\omega^7] + b^7_{16}[\omega^1\omega^6] + b^7_{18}[\omega^1\omega^8] + b^7_{68}[\omega^6\omega^8];\\
D\omega^8 &= [\omega^5, \omega^1 + \omega^2] + [\theta^8_8\omega^8] + b^8_6[\omega^1 - \omega^2, \omega^6 + \omega^7],
\end{aligned} \tag{13}$$

where $a^1_{16} = a^1_{28} = 1$; $a^1_{26} = -(\gamma + 1)/2$; $a^1_{27} = (\gamma - 3)/2$; b^i_{jk} are some-

expressions in terms of $x, t, u, p, \rho, u_x, \rho_x, p_x$; the forms $\omega^1, \omega^2, \omega^6, \omega^7, \omega^8$ are the principal forms determining a point of the integral manifold; the forms $\omega^3, \omega^4, \omega^5$ are secondary forms, depending on the differentials of the parametric derivatives $du_x, dp_x, d\rho_x$; $\omega, \theta^6_6, \theta^7_7, \theta^8_8$ are linear combinations of the principal forms.

Let us set some of the coefficients b^i_{jk} equal to zero by means of a canonization. For this purpose introduce the forms:

$$\bar{\omega}^6 = \omega^6 + l\omega^1; \quad \bar{\omega}^7 = \omega^7 + m\omega^2; \quad \bar{\omega}^8 = \omega^8 + n(\omega^1 + \omega^2), \tag{14}$$

where l, m, n are certain, as yet undetermined, coefficients. In the forms $\omega^1, \omega^2, \bar{\omega}^6, \bar{\omega}^7, \bar{\omega}^8$ the system (13) will retain its form, except that b^i_{jk} will pass into $\bar{b}^i_{jk}$, where, for example:

$$\bar{b}^6_{27} = b^6_{27} + nb^6_{78} + la^1_{27},$$

$$\bar{b}^6_{28} = b^6_{28} - mb^6_{78} + l, \tag{15}$$

$$\bar{b}^7_{16} = b^7_{16} + nb^6_{78} - ma^1_{27}.$$

Choose l, m, n so as to make $\bar{b}^6_{27}, \bar{b}^6_{28}, \bar{b}^7_{16}$ vanish. The system of linear nonhomogeneous equations (15) in l, m, n has, as is not hard to verify, determinant equal to $(3 - \gamma)$. Consequently, if $\gamma \neq 3$, then l, m, n are uniquely determined from the system (15). Substituting these expressions for all l, m, n into (14), we obtain:

$$\bar{\omega}^6 = \frac{1}{4a}\left(du - \frac{dp}{\rho a}\right); \quad \bar{\omega}^7 = \frac{1}{4a}\left(du + \frac{dp}{\rho a}\right); \quad \bar{\omega}^8 = \frac{1}{4\gamma} d \ln\left(\frac{p}{\rho^\gamma}\right). \tag{16}$$

Note that the forms $\bar{\omega}^6, \bar{\omega}^7, \bar{\omega}^8$ are dimensionless. The system of exterior differential equations for $\omega^1, \omega^2, \bar{\omega}^6, \bar{\omega}^7, \bar{\omega}^8$ has the form

$$D\bar{\omega}^i = c^i_{jk}\,[\bar{\omega}^j \bar{\omega}^k],$$

where c^i_{jk} are constants (here, for uniformity, we have introduced the notation $\omega^1 = \bar{\omega}^1,\ \omega^2 = \bar{\omega}^2$).

This means that the forms $\omega^1, \omega^2, \bar{\omega}^6, \bar{\omega}^7, \bar{\omega}^8$ are basic invariant forms of a certain group g ([2]). The group g is a subgroup of the fundamental group G admitted by the system (1)–(3). It is also not difficult to show that for $\gamma \neq 3$ the group G is 6-parameter, while for $\gamma = 3$ the group is enlarged to a 7-parameter one. This was first established in ([1]).

It can be shown that for $\gamma \neq 3$ the basic invariant forms of the group G_6 are as follows:

$$\bar{\omega}^1 = \lambda[dx - (u - a)dt]; \quad \bar{\omega}^2 = \lambda[dx - (u + a)dt];$$

$$\bar{\omega}^6 = \frac{1}{4a}\left(du - \frac{dp}{\rho a}\right); \quad \bar{\omega}^7 = \frac{1}{4a}\left(du + \frac{dp}{\rho a}\right); \tag{17}$$

$$\bar{\omega}^8 = \frac{1}{4\gamma} d\ln\left(\frac{p}{\rho^\gamma}\right); \quad \bar{\omega}^9 = \frac{d\lambda}{\lambda},$$

where $\bar{\omega}^9$ is a new form containing the new variable λ. Let the entropy be constant in the flow. Then $p/\rho^\gamma = \text{const}$, and only 5 forms from (17) remain. Consider now the equalities (14) on the integral manifold, where $\omega^6 = \omega^7 = \omega^8 = 0$:

$$\frac{1}{4a}\left(du - \frac{dp}{\rho a}\right) = l[dx - (u-a)dt]; \tag{18^1}$$

$$\frac{1}{4a}\left(du + \frac{dp}{\rho a}\right) = m[dx - (u+a)dt]; \tag{18^2}$$

$$n = 0. \tag{18^3}$$

If we move along a characteristic of the first family, then from (18^1) it immediately follows that $I^- = \text{const}$; analogously, from (18^2) we obtain that along a characteristic of the second family $I^+ = \text{const}$.

As is known ([3]), if the system of forms ω^i satisfies equations (4^*), then the integrals of the system $\omega^i = 0$ may be regarded as coordinates in the parameter space of the group G. From (17) it is clear (assuming that $S = \text{const}$) that these coordinates in the group space are: x, t, λ, I^+, I^-. Thus we obtain:

Theorem. *The Riemann invariants I^+, I^- may be regarded as coordinates in the parameter space of the group G_5 leaving the original system* (1)–(3) *invariant.*

The method used in this work for studying the group properties of the system (1)–(3), by passing from the original forms ω^i to the forms $\tilde{\omega}^i$, which are basic invariant forms of a certain group, is a concretization of the method applied by A. M. Vasil' ev to the study of an arbitrary system $F_i(x, t, u, p, \rho, u_x, u_t, \dots, \rho_t) = 0$ $(i = 1, 2, 3)$ ([5,6]).

In conclusion, I consider it my pleasant duty to express my gratitude to A. M. Vasil' ev for critical comments and attention to the work.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.