ON THE UNIQUE SOLVABILITY OF PLANE PROBLEMS OF GAS DYNAMICS WITH FREE BOUNDARIES

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Abstract

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ON THE UNIQUE SOLVABILITY OF PLANE PROBLEMS OF GAS DYNAMICS WITH FREE BOUNDARIES

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In the present paper we prove the existence and uniqueness of subsonic potential gas flows for a broad class of problems of gas dynamics with free boundaries, among which, for example, the following problems may be indicated:

1. Problems of flow with separation according to Kirchhoff' s scheme past a given (unknown) curvilinear obstacle by an unbounded gas stream and by a gas stream in a channel with a given (unknown) shape of the walls. The shape of the unknown part of the boundary L_z of the flow domain is determined by the prescribed value on it of the speed $q = q(x)$ (on the jets $q = \text{const}$).
2. The inverse and mixed problem of determining the shape of an airfoil profile L_z from the chord diagram, when on the unknown part of the profile (or on all of it in the inverse problem) the value of the speed $q = q(x)$ is prescribed. In this problem the solution may also correspond to a nonclosed contour L_z.

The basis of the method we propose for investigating problems of gas dynamics is the apparatus of generalized analytic functions, well developed by I. N. Vekua ([1]).

For an incompressible fluid in problem 1, which has already become classical, the principal results were obtained by M. A. Lavrent' ev ([2]), Leray ([3]), and a number of other authors. Problems 1 and 2 in the case of an incompressible fluid were investigated in the author' s papers ([4],[5]).

Let $\theta = \operatorname{arctg} \varphi_y/\varphi_x$ be the angle of inclination of the flow velocity ($\theta_\infty = 0$), and suppose that on the prescribed parts of the boundary L_z, $\theta(x) \in C^1_\alpha$ and

$$|\theta(x)| < \pi/2 - \varepsilon_0 \qquad (\varepsilon_0 > 0). \tag{1}$$

On the sought parts of the boundary L_z, where the value of the speed $q(x) \in C^1_\alpha$ is prescribed, it is assumed that the inequality $|\ln q(x)| < \infty$ is satisfied, which is equivalent to

$$q^*(x) = \int_1^{q(x)} \frac{\sqrt{1-\mathrm{M}^2}}{q}\, dq < N < \infty, \tag{2}$$

where $1 - \mathrm{M}^2 = \dfrac{1}{\rho}\dfrac{d(q\rho)}{dq}$ and $\rho = \rho(q)$ is the density.

In view of conditions (1) and (2), at the branch points of the flow the boundary L_z has a "cusp" directed toward the flow, and the channel walls are parallel at infinity.

As is known ([6]), the equations of gas dynamics in the case of a plane potential flow can be written in the form

$$\rho(q)\varphi_x = \psi_y; \qquad \rho(q)\varphi_y = -\psi_x,$$

where $q = (\varphi_x^2 + \varphi_y^2)^{1/2}$ is the magnitude of the flow velocity.

The flow is assumed throughout to be subsonic ([6]), i.e., the ellipticity of the system of equations of gas dynamics is assumed for all q $(1 - \mathrm{M}^2 > 0)$. From this assumption, in particular, it follows that fulfillment of inequalities (1), (2) on the entire boundary L_z entails their equu—

numbered fulfillment throughout the domain D_z. In the plane of the variables x, ψ, for the functions $q^*(x,\psi)$ and $\theta(x,\psi)$ one obtains the complex equation

$$\omega_{\bar\tau} - \mu_1(\omega)\omega_\tau = 0, \tag{3}$$

where $\omega = q^* - i\theta,\ \tau = x + i\psi$,

$$\mu_1(\omega) = \frac{\rho q - \cos\theta\sqrt{1-M^2} + i\sin\theta}{\rho q + \cos\theta\sqrt{1-M^2} - i\sin\theta}.$$

The system of equations corresponding to the complex equation (3) is elliptic provided inequalities (1) and (2) are fulfilled on the entire boundary L_z. In the case of incompressibility of the fluid, $\rho = 1,\ M = 0,\ \omega = \ln q - i\theta$ and $\mu_1(\omega) = (e^\omega - 1)/(e^\omega + 1)$ ([4,5]).

In problems 1 and 2, in the plane $\tau = x + i\psi$, the image D_τ of the flow domain D_z is, obviously, known; moreover, the fulfillment of inequalities (1), (2) on the entire boundary ensures the quasiconformality of the mapping $z = z(\tau)$. In jet problems this is the plane or a strip with a cut along a semi-infinite interval of the real axis; in problems of determining an airfoil profile it is the plane with a cut along a finite segment of the real axis.

Map conformally D_τ onto the interior of the circle $|\zeta| < 1$ in such a way that the image of the unknown part of the boundary L_z is mapped into the upper semicircle, and that of the prescribed part into the lower one. Then L_z is completely unknown, and the normalization of the mapping is arbitrary. Then equation (3) is transformed into the form

$$\omega_{\bar\zeta} - \mu(\omega, \zeta)\omega_\zeta = 0, \tag{4}$$

where $\mu(\omega, \zeta) = \mu_1(\omega)\exp\{-2i\arg(d\tau/d\zeta)\}$. For a completely unknown boundary L_z, for equation (4) we obtain the Dirichlet boundary-value problem

$$\operatorname{Re}\omega(e^{i\gamma}) = q^*[x(\gamma)], \quad \gamma \in [0, 2\pi]; \qquad \operatorname{Im}\omega(1) = 0; \tag{5}$$

but if a part of L_z is prescribed, we obtain the mixed boundary-value problem

$$\operatorname{Re}\omega(e^{i\gamma}) = q^*[x(\gamma)], \quad \gamma \in [0, \pi]; \qquad \operatorname{Im}\omega(e^{i\gamma}) = -\theta[x(\gamma)],$$

$$\gamma \in [\pi, 2\pi]. \tag{6}$$

From the conditions on the functions $q^*(x)$ and $\theta(x)$ it follows that $\omega(z) \in C^1_\alpha$ on L_z, except, possibly, for the junction points of the unknown and prescribed parts of L_z, in neighborhoods of which $\omega(z) \in C_\alpha$. The function $\omega(\zeta)$ has an analogous property. From this, in particular, follows the fulfillment of inequality (1) on the entire boundary L_z.

We shall prove the uniqueness of the solutions of problems 1 and 2 satisfying inequalities (1) and (2) on the entire boundary L_z. For this it is obviously sufficient to show the uniqueness of the function $\omega(\zeta)$ solving problem (4), (5) or (4), (6) under the fulfillment of the inequality $|\mu(\omega, \zeta)| \le \mu_0 < 1$. Suppose there are two such functions ω^1 and ω^2. Then $\omega = \omega^1 - \omega^2$ satisfies homogeneous boundary conditions and the equation

$$\omega_{\bar\zeta} - \mu(\omega^1, \zeta)\omega_\zeta + B(\zeta)\omega = 0, \tag{7}$$

where

$$B(\zeta) = \omega^2_\zeta(\omega^2 - \omega^1)^{-1}[\mu(\omega^2, \zeta) - \mu(\omega^1, \zeta)]$$

for $\omega^1 - \omega^2 \ne 0$, and $B(\zeta) = 0$ for $\omega^1 = \omega^2$. Since $\omega \in L_p$ $(p > 2)$, and the function $\mu(\omega, \zeta)$ is differentiable with respect to the argument ω, it follows that

$B(\zeta) \in L_p$ $(p > 2)$. Let $\chi = \chi(\zeta)$ be a homeomorphism of the circle K $(|\zeta| < 1)$ onto the circle $|\chi| < 1$, satisfying the Beltrami equation $\chi_{\bar\zeta} - \mu(\omega^1, \zeta)\chi_\zeta = 0$ and leaving the points ± 1 in place. Denote

$$B(\zeta)\,[\bar\chi_{\bar\zeta} - \mu\chi_\zeta]^{-1} = c(\chi) \in L_p \ (p > 2)$$

and

$$-\frac{1}{\pi}\iint_K \left[\frac{c(t)}{t-\chi} + \frac{\chi c(\bar t)}{1-\chi\bar t}\right] dK = \widetilde{T}_1 c.$$

Then any solution of equation (7) can be represented in the form

$$\omega = \Phi(\chi)\exp\{\widetilde{T}_1 c\} \quad (^1),$$

where $\Phi(\chi)$ is an analytic function.

Since $\operatorname{Im}\widetilde{T}_1 c = 0$ on the circle $|\chi| = 1$, the analytic function $\Phi(\chi)$ satisfies the homogeneous problem (5) or (6) and is bounded in the closed disk $|\chi| \leqslant 1$; consequently, $\Phi(\chi) \equiv 0$, whence $\omega = \omega^1 - \omega^2 \equiv 0$.

Let us now prove the existence of solutions in problems 1 and 2. We shall assume that, on the unknown part L_z, one of the conditions $q = \text{const}$ or $q = q(x)$ is fulfilled, with $|dq/dx| < \infty$, $q(x) \in C_{\alpha,1}$, for which satisfaction of inequalities (1) and (2) on the whole boundary L_z is shown. We include problems (4), (5) and (4), (6) in a one-parameter family of problems by introducing in conditions (5) and (6), before $q^*[x(\gamma)]$ and $-\theta[x(\gamma)]$, the multiplier λ. The value $\lambda = 0$ corresponds to the trivial solution $\omega(\zeta) \equiv 0$, and $\lambda = 1$ to the solution of the original boundary-value problems.

Consider the operators

$$T_1 f = \frac{1-\zeta}{\pi}\iint_K \left[\frac{f(t)}{(t-1)(t-\zeta)} + \frac{\overline{f(t)}}{(1-\bar t)(1-\zeta t)}\right] dK,$$

$$T_2 f = -\frac{R(\zeta)}{\pi}\iint_K \left[\frac{f(t)}{R(t)(t-\zeta)} - \frac{\overline{f(t)}}{\overline{R(t)}(1-\zeta\bar t)}\right] dK,$$

where $f(t) \in L_p$, and $R(\zeta) = \sqrt{i(\zeta^2-1)}$. The well-studied properties, due to I. N. Vekua, of the operators $\widetilde{T}_1$ and T_1 were used by us as the basis for the construction and study of the operator T_2 $(^5)$. The operators T_1, T_2 are completely continuous in L_p $(p > 2)$; $\partial T_i f/\partial\bar\zeta = f$ $(i = 1, 2)$ for any function $f \in L_p$; $T_i f$ satisfy, respectively, the homogeneous conditions (5) and (6), while the operators $S_i f = \partial T_i f/\partial\zeta$ are bounded in L_p and $\|S_i\|_{L_2} = 1$. We shall seek the solution of the family of boundary-value problems (4), (5) and (4), (6) in the form: $\omega(\zeta,\lambda) = \lambda\Phi_i(\zeta) + T_i f$, where $f \in L_p$ $(p > 2)$ and $\Phi_i(\zeta)$ $(i = 1, 2)$ are analytic functions satisfying, respectively, the boundary conditions (5) and (6). Then, for the function $f(\zeta)$, the singular integral equations are obtained

$$f - \mu(\zeta, \lambda\Phi_i + T_i f)S_i f - \lambda\mu(\zeta, \lambda\Phi_i + T_i f)\, d\Phi_i/d\zeta = 0. \tag{8}$$

Denote by G the set of functions $g(\zeta)$ for which $\Omega_i = \lambda\Phi_i + T_i g$ satisfy, in the disk $|\zeta| \leqslant 1$, the conditions $|\operatorname{Re}\Omega_i| < N < \infty$ and $|\operatorname{Im}\Omega_i| < \pi/2 - \varepsilon_0$. Let $g(\zeta)$ and $f(\zeta)$ be arbitrary functions of the set G. Then

$$\|\mu(\eta, \Omega_i)S_i f\|_{L_p} \leqslant \mu_0[1 + \delta(p)]\|f\|_{L_p},$$

where $\delta(p) \to 0$ as $p \to 2$, and therefore $\mu_0(1 + \delta) < 1$ for p sufficiently close to 2.

Thus equation (8) can be written in the form

$$f - A_i(\Omega_i, \lambda) = 0, \tag{9}$$

where $A_i(\Omega_i, \lambda) = \lambda(E - \mu S_i)^{-1}\mu(d\Phi_i/d\zeta)$. From the continuity of the operators A_i in $\Omega_i = \lambda\Phi_i + T_i g$ and the complete continuity of the operators T_i, it follows that $A_i(\lambda\Phi_i + T_i g, \lambda)$ is completely continuous in g for any fixed λ, and continuous in λ for arbitrary fixed $g \in G$. Thus we are in the conditions of applicability to the operators A_i of the Leray–Schauder method ([7]).

It is evident that the method proposed by us also yields existence and uniqueness theorems for problems of flow in channels with completely prescribed wall shape and of flow around a prescribed profile.

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REFERENCES

[1] I. N. Vekua, *Generalized analytic functions*, Moscow, 1959.
[2] M. A. Lavrent' ev, *Matem. sborn.*, 4 (46), 391 (1938).
[3] J. Leray, *Comm. Math. Helv.*, 8, 148 (1935).
[4] V. N. Monakhov, Proceedings of the seminar on inverse boundary-value problems, issue 1, Kazan, 1964.
[5] V. N. Monakhov, Proceedings of the seminar on inverse boundary-value problems, issue 2, Kazan, 1964.
[6] L. Bers, *Mathematical problems of subsonic and transonic gas dynamics*, IL, 1961.
[7] J. Leray, J. Schauder, *Ann. de l' Ecole Norm. Sup.*, 13, 1934; *UMN*, 1, No. 3–4 (1946).