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**Abstract**

**Full Text**

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## AN EXCEPTIONAL CASE OF CONVOLUTION-TYPE INTEGRAL EQUATIONS IN THE CLASS OF GENERALIZED FUNCTIONS

*(Presented by Academician N. I. Muskhelishvili, 31 VIII 1964)*

The following convolution-type integral equations in the class of generalized functions (g.f.) are considered: the equation “with two kernels”

$$\lambda f(x) + \frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(x-t)f(t) dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 k_2(x-t)f(t) dt = g(x),$$

$$-\infty < x < \infty, \tag{1}$$

and the “paired” equations

$$\lambda_1 f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_1(x-t)f(t) dt = g(x), \quad x > 0,$$

$$\lambda_2 f(x) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty k_2(x-t)f(t) dt = g(x), \quad x < 0. \tag{2}$$

Equations (1), (2) are investigated by reducing them to a problem of linear conjugation ( $\hat{1}$ ), whose coefficients are allowed zeros of integral order at a finite number of points of the contour, including the infinitely distant point.

In the class  $L_2(-\infty, \infty)$  an analogous case was considered in ( $\hat{2}$ ).

1. We introduce the following notation. Let  $n \geq 0, m \geq 0$  be integers.  $L_2(-n, -m)$  is the space of basic functions  $\varphi(x)$ , defined on the real axis, differentiable  $m$  times, with norm

$$\|\varphi\| = \left( \int_{-\infty}^\infty \left| \left( \frac{d}{dx} + 1 \right)^m \{(x+i)^n \varphi(x)\} \right|^2 dx \right)^{1/2}. \tag{3}$$

Then  $L_2(n, m)$  is the space of g.f.  $f$  over the space  $L_2(-n, -m)$ , i.e. of linear functionals continuous with respect to the norm (3).

The Cauchy operator  $S$ , acting in  $L_2(0, m)$  by the formula

$$(Sf, \varphi) = \left( f, -\frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{\varphi(t) dt}{t-x} \right), \quad f \in L_2(0, m), \quad \varphi \in L_2(0, -m),$$

is linear and maps  $L_2(0, m)$  onto itself. By  $L_2^\pm(0, m)$  we denote the subspaces of the space  $L_2(0, m)$  of g.f.  $f^\pm$  satisfying the equations  $f^\pm = \pm S f^\pm$ .

By  $L_{2\pm}(n, m)$  we denote the subspaces of the space  $L_2(n, m)$  of g.f.  $f_\pm$ . The Fourier operator  $V$  establishes a homeomorphism between the spaces  $L_2(n, m)$  and  $L_2(m, n)$ . In this case the following properties hold:

$$(x \pm i)^{-m} F^\pm = (x \pm i)^{-m} V f_\pm \in L_2^\pm(0, n),$$

where  $f_\pm \in L_{2\pm}(n, m)$ ,  $F^\pm \in L_2^\pm(m, n) = V[L_{2\pm}(n, m)]$ .

Let  $r$  be an integer. By  $L_2(r, n, \rho(x))$  we denote the space of g.f.  $G$  over the basic space  $L_2(-r, -n, \rho(x))$  of functions  $\psi(x)$ ,

defined on the real axis, differentiable  $n$  times, with norm

$$\|\psi\| = \left( \int_{-\infty}^{\infty} \rho(x) \left| \left( \frac{d}{dx} + 1 \right)^n \{(x+i)^r \psi(x)\} \right|^2 dx \right)^{1/2},$$

where  $\rho(x)$  is a nonnegative function bounded on the real axis.

**2. The linear conjugation problem.** Find generalized functions  $F^+ \in L_2^+(m, n)$  and  $F^- \in L_2^-(m, n)$  satisfying the relation

$$\frac{\prod_{k=1}^p (x - a_k)^{\alpha_k}}{(x+i)^{\alpha+\alpha_0}} \Omega(x) F^+ = \frac{\prod_{j=1}^q (x - b_j)^{\beta_j}}{(x+i)^{\beta+\beta_0}} T(x) F^- + G, \quad (4)$$

where  $\text{Im } a_k = \text{Im } b_j = 0$ ,  $1 \leq k \leq p$ ,  $1 \leq j \leq q$ ;  $\alpha_k$ ,  $0 \leq k \leq p$ , and  $\beta_j$ ,  $0 \leq j \leq q$ , are positive integers, and moreover

$$\sum_{k=1}^p \alpha_k = \alpha, \quad \sum_{j=1}^q \beta_j = \beta, \quad \max_{\substack{1 \leq k \leq p \\ 1 \leq j \leq q}} \{\alpha_k, \beta_j\} = n_0 \leq n;$$

$\Omega(x)$  and  $T(x)$  are  $n$ -times continuously differentiable functions, nonzero on the real axis,  $\Omega^{(n)}(x) \in H_\mu$ ,  $T^{(n)}(x) \in H_\mu$ ,  $0 < \mu < 1$ . The free term of the problem is  $G \in L_2(r, n - n_0, \rho_1(x))$ , where  $r \leq m - m_1 = s$ ,  $m_1 = \max\{\alpha_0, \beta_0\}$ ,

$$\rho_1(x) = \left| \frac{\prod_{k=1}^{p_1} (x - \tilde{a}_k)}{(x+i)^{p_1}} \right|^{\varepsilon_1}; \quad 0 < \varepsilon_1 < 1;$$

$\tilde{a}_k$  are those among  $a_k, b_j$  for which  $\alpha_k = n_0, \beta_j = n_0$ .

We shall call the number  $\chi = \chi_1 + \beta + \beta_0 + s$  the index of problem (4), where

$$\chi_1 = \frac{1}{2\pi i} [\ln T(x) - \ln \Omega(x)]_{\pm\infty}^{\pm\infty}.$$

Let us formulate the final result.

**Theorem.** If  $\chi > 0$ , the homogeneous problem (4) is solvable and has  $\chi$  linearly independent solutions; if  $\chi \leq 0$ , it has no solutions distinct from the identically zero one. Its general solution has the form

$$\begin{aligned} F^+ &= (x+i)^{\alpha_0+s} X^+(x) \frac{P_{\chi-1}(x)}{(x+i)^{\chi-1}} \sum_{k=1}^p \sum_{s=0}^{\alpha_k-1} A_{ks} \delta^{(s)+}(x-a_k), \\ F^- &= (x-i)^{\beta_0+s} X^-(x) \frac{P_{\chi-1}(x)}{(x-i)^{\chi-1}} \sum_{j=1}^q \sum_{l=0}^{\beta_j-1} B_{jl} \delta^{(l)-}(x-b_j). \end{aligned} \quad (5)$$

Here  $P_{\chi-1}(x)$  is a polynomial of degree  $\chi-1$  with arbitrary coefficients;  $A_{ks}$  and  $B_{jl}$  are completely determined constants;  $X^\pm(x)$  are the boundary values of the canonical function of the homogeneous problem with coefficient  $((x+i)/(x-i))^{\chi_1} T(x)/\Omega(x)$ .

The nonhomogeneous problem (4) is solvable for  $\chi \geq 0$  and has  $\chi$  linearly independent solutions if  $\chi > 0$ ; for  $\chi = 0$  its solution is unique. The general solution of the nonhomogeneous problem is the sum of the general solution (5) of the homogeneous problem and a particular solution of the nonhomogeneous problem. The latter has the form

$$\begin{aligned} (Y^+, \Phi) &= \left( (x+i)^{\alpha_0+s} X^+(x) G_1^+, \frac{(x+i)^\alpha \Phi(x) - Q(x)}{\prod_{k=1}^p (x-a_k)^{\alpha_k}} \right), \\ (Y^-, \Phi) &= \left( (x-i)^{\beta_0+s} \left( \frac{x+i}{x-i} \right)^\chi X^-(x) G_1^-, \frac{(x+i)^\beta \Phi(x) - Q_1(x)}{\prod_{j=1}^q (x-b_j)^{\beta_j}} \right), \end{aligned} \quad (6)$$

where  $Q(x)$  is the Hermite interpolation polynomial (see (4)) of degree  $a-1$  for the function  $(x+i)^\alpha \Phi(x)$  with interpolation nodes  $a_k$  of multiplicity  $\alpha_k$ ;  $Q_1(x)$  is the analogous interpolation polynomial for the function  $(x+i)^\beta \Phi(x)$ ;  $G_1 = [(x+i)^s X^+(x) \Omega(x)]^{-1} G$ ,  $G_1^\pm = \pm \frac{1}{2} G_1 + S G_1$ .

If  $\chi < 0$ , then the nonhomogeneous problem is, generally speaking, unsolvable; for its solvability the following  $|\chi|$  additional conditions imposed on the right-hand side  $G$  are necessary and sufficient:

$$\left( [X^+(x)\Omega(x)]^{-1}G, \frac{1}{(x+i)^{k+s}} \right) = 0, \quad k = 1, 2, \dots, |\chi|.$$

When these conditions are satisfied, the solution of the problem is unique and is given by formulas (6).

### 3. Integral equations of convolution type

The solutions of equations (1) and (2) are sought in the space  $L_2(n, m)$ . The following restrictions are imposed on the convolution kernels  $k_j(x)$ :  $K_j(x) = V[k_j(x)] \in C^{(n)}$ , and  $K_j^{(n)}(x) \in H_\mu$  ( $0 < \mu < 1$ ),  $j = 1, 2$ .

The conventional notation (1) for  $\lambda = 0$  is understood in the following sense:

$$\left( f_+, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_1(t-x)\varphi(t) dt \right) - \left( f_-, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2(t-x)\varphi(t) dt \right) = (g, \varphi) \quad (7)$$

for any test function  $\varphi \in L_2(-n, -m)$ ,  $f_\pm$  being unknown generalized functions from the space  $L_{2\pm}(n, m)$ .

Taking the Fourier transform of equation (7), we arrive at the following boundary-value problem in the space  $L_2(m, n)$ :

$$K_1(x)F^+ = K_2(x)F^- + G. \quad (8)$$

Let the functions  $K_j(x)$  admit the representations

$$K_1(x) = \frac{\prod_{k=1}^p (x - a_k)^{\alpha_k}}{(x+i)^{\alpha+\alpha_0}} \Omega(x), \quad K_2(x) = \frac{\prod_{j=1}^q (x - b_j)^{\beta_j}}{(x+i)^{\beta+\beta_0}} T(x),$$

where  $a_k, b_j, \alpha_k, \beta_j, \Omega(x), T(x)$  satisfy the conditions of problem (4). Then, for  $G \in L_2(r, n - n_0, \rho(x))$ ,  $r \leq s$ , the results of the theorem formulated above are valid for problem (8), and its solution can be found from formulas (5), (6). Once the general solution  $F^\pm$  of problem (8) has been found, the general solution of equation (7) is given by the formula  $f_\pm = V^{-1}F^\pm$ .

Equations (2), in exact notation for  $\lambda_1 = \lambda_2 = 0$ , have the form

$$\left( f, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_1(t-x)\varphi_+(t) dt \right) = (g, \varphi_+),$$

$$\left( f, \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} k_2(t-x)\varphi_-(t) dt \right) = (g, \varphi_-), \quad (9)$$

where  $\varphi_{\pm} \in L_{2\pm}(-n, -m)$ , i.e., they are test functions from the space  $L_2(-n, -m)$  equal to zero for  $x < 0$  and  $x > 0$ , respectively. Extending equalities (9) to the whole  $x$ -axis and applying the Fourier transform to them, we obtain the equivalent system of equations

$$K_1(x)F = G^+ + \Psi^-, \quad K_2(x)F = G^- + \Psi^+, \quad (10)$$

where  $\Psi^+ \in L_2^+(m - \beta_0, n)$ ,  $\Psi^- \in L_2^-(m, \alpha_0, n)$  are new unknown generalized functions.

Let the coefficients  $K_j(x)$  have the form

$$K_1(x) = \frac{A(x)C(x)\Omega(x)}{(x+i)^{\alpha+\gamma+\alpha_0}}, \quad K_2(x) = \frac{B(x)C(x)T(x)}{(x+i)^{\beta+\gamma+\beta_0}},$$

where

$$A(x) = \prod_{k=1}^p (x - a_k)^{\alpha_k}, \quad B(x) = \prod_{j=1}^q (x - b_j)^{\beta_j}, \quad C(x) = \prod_{l=1}^h (x - c_l)^{\gamma_l};$$

$a_k, b_j, \alpha_k, \beta_j, \Omega(x), T(x)$  satisfy the previous conditions; moreover,

$$a_k \neq b_j, \quad \text{Im } c_l = 0, \quad \gamma_l \geq 0 \text{ are integers,} \quad \sum_{l=1}^h \gamma_l = \gamma, \quad \max_l \{\gamma_l\} \leq n_0^*.$$

The boundary-value problem by which the generalized functions  $\Psi^{\pm}$  are determined has the form

$$\frac{A(x)\Omega(x)}{(x+i)^{\alpha-\beta_0}} \Psi^+ = \frac{B(x)T(x)}{(x+i)^{\beta-\alpha_0}} \Psi^- + G_1, \quad (11)$$

where

$$G_1 = \frac{B(x)T(x)}{(x+i)^{\beta-\alpha_0}} G^+ - \frac{A(x)\Omega(x)}{(x+i)^{\alpha-\beta_0}} G^-,$$

and is analogous to problem (4). If

$$G^+ \in L_2(r_1, n - n_0, \rho_2(x)), \quad G^- \in L_2(r_2, n - n_0, \rho_2(x)), \quad r_1 \leq m - \alpha_0,$$

$$r_2 \leq m - \beta_0, \quad \text{where} \quad \rho_2(x) = \left| \frac{\prod_{k=1}^{p_2} (x - \tilde{b}_k)}{(x + i)^{p_2}} \right|^{\varepsilon_2}, \quad 0 < \varepsilon_2 < 1,$$

$\tilde{b}_k$  are those among  $a_k, b_j, c_l$  for which  $\alpha_k = n_0, \beta_j = n_0, \gamma_l = n_0$ , then for  $\chi \geq 0, \chi = \chi_1 + \beta + m - \alpha_0$  is the index of the problem, it is solvable in the indicated classes of generalized functions.

Solving successively the boundary-value problem (11) and the system (10), by the inverse Fourier transform of the solution  $F$  of the latter one can find the solution  $f = V^{-1}F$  of the original system of integral equations (9).

Let us note that the solution  $F$  of system (10) is determined up to a functional

$$\sum_{l=1}^h \sum_{j=0}^{\gamma_l-1} C_{lj} \delta^{(j)}(x - c_l),$$

where  $C_{lj}$  are arbitrary constants.

We formulate the final result for “even” equations. The homogeneous system of equations (9), when the functions  $K_1(x)$  and  $K_2(x)$  have common zeros  $c_l$ , respectively of multiplicities  $\gamma_l, 1 \leq l \leq h$ , is unconditionally solvable in the space  $L_2(n, m)$ ; for  $\chi > 0$  it has  $\chi + \gamma$  linearly independent solutions, while for  $\chi \leq 0$  the number of its solutions is equal to

$$\gamma = \sum_{l=1}^h \gamma_l **.$$

The nonhomogeneous system (9), under the restrictions indicated above on the right-hand side, is unconditionally solvable in the space  $L_2(n, m)$  if  $\chi \geq 0$ ; for its solvability in the case  $\chi < 0$  the following  $|\chi|$  additional conditions imposed on the right-hand side are necessary and sufficient:

$$\left( \left( \frac{x-i}{x+i} \right)^{\chi_1} \frac{B(x)}{(x+i)^{\beta+m-\alpha_0}} \frac{1}{X^-(x)} G^+ - \frac{A(x)}{(x+i)^{\alpha+m-\beta_0}} \frac{1}{X^+(x)} G^- , \frac{1}{(x+i)^k} \right) = 0,$$

$$k = 1, 2, \dots, |\chi|.$$

The well-known Wiener-Hopf equation of the first kind, which is a special case ( $k_2(x) \equiv 0$ ,  $G^- \equiv 0$ ) of the equations considered, in one subclass of generalized functions from the space  $L_2(n, m)$  was studied in work (5).

The normal case of the problem of linear conjugation and of equations (1), (2) in the space  $L_2(n, m)$  was investigated by Yu. I. Cherskii (3), under whose supervision the present work was carried out.

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\* Equalities  $c_{l_0} = a_{k_0}$  ( $c_{l_0} = b_{j_0}$ ) are admissible, but it is then required that

$$\gamma_{l_0} + \alpha_{k_0} \leq n_0 \quad (\gamma_{l_0} + \beta_{j_0} \leq n_0).$$

\*\* In the absence of common zeros of the coefficients  $K_j(x)$ ,  $j = 1, 2$ , the number of linearly independent solutions of the homogeneous system is determined by the index  $\chi$ .

*Note: Figure translations are in progress. See original paper for figures.*

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