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## Abstract

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# A GENERAL SCHEME FOR SOLVING BOUNDARY-VALUE PROBLEMS IN THE SPACE OF GENERALIZED FUNCTIONS

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The Riemann boundary-value problem in the space of generalized functions has been studied by many authors. The formulation of the problem apparently belongs to O. S. Parasyuk ([1]), who considered the problem in the case when the contour is the real axis. Earlier, Köthe ([2]) considered a special case of the Riemann problem—the problem of decomposing a generalized function into the difference of boundary values of functions analytic inside and outside an analytic curve (for bibliography, see ([3])). In the present note a general scheme is set forth for solving boundary-value problems in the space of generalized functions, applicable both to the Riemann problem and to the problem of linear conjugation with shift, the Carleman problem ([4]) and its various generalizations ([5]), as well as to the Hilbert problem (([6], p. 236)).

1. Let $L$ be a closed simple curve dividing the plane into two domains $D^+$ (the interior of $L$) and $D^-$ (the exterior of $L$). We shall assume that the function $z = \Psi(w)$ ($\Psi(\infty) = \infty$, $\Psi'(\infty) > 0$), mapping the exterior of the unit circle $|w| > 1$ onto the exterior of $L$, is continuous in the closed domain $|w| \geq 1$ together with derivatives of arbitrary order, and moreover $\Psi'(e^{i\theta}) \neq 0$. Consider on $L$ the basic space $S$, consisting of infinitely differentiable functions $\varphi(t)$ of the points $t = x + iy$ of the contour $L$. We introduce convergence in $S$ as follows: the sequence $\{\varphi_k\}_1^\infty$ is called convergent to zero, and we write $\varphi_k \to 0$ as $k \to \infty$, if for every fixed $s$ the sequences $d^s\varphi_k(t)/dt^s$ converge uniformly. Linear functionals $(f, \varphi)$ that are continuous in the sense of the convergence introduced in $S$ will be called generalized functions (g.f.). In ([3]) the subspaces $S^+$ and $S^-$ of the space $S$ were introduced, and g.f. of plus type $f^+$ and minus type $f^-$ were defined. It was proved that g.f. of plus type are analytically continued into $D^+$, and g.f. of minus type into $D^-$. Recall that $(f^+, \varphi^+) = 0$ for all $\varphi^+ \in S^+$ and $(f^-, \varphi^-) = 0$ for all $\varphi^- \in S^-$.
2. Consider the following problems:

a) the Riemann problem

$$f^+ = G(t)f^- + g, \quad t \in L;$$

b) the problem of linear conjugation with shift

$$f^+[\alpha] = G(t)f^- + g, \quad t \in L; \tag{1}$$

c) the Carleman problem

$$f^+ = G(t)f^+[\alpha] + g, \quad t \in L;$$

d) the Hilbert problem*

$$\mathrm{Re}[f^+/G(t)] = g, \quad t = e^{i\theta},\ 0 \le \theta \le 2\pi.$$

Here $G(t) \in S$ $(G(t) \neq 0)$**, $\alpha(t) \in S$ $(\alpha'(t) \neq 0)$, and the g.f. $g$ are given; $f^+, f^-$ are the unknown g.f. The shift of a generalized function is defined by the equality

$$(f[\alpha], \varphi(t)) = (f, \varphi[\beta(t)] \cdot \beta'(t)),$$

where $\alpha[\beta(t)] \equiv t$, and it is assumed that

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* For an explanation of the notation of boundary condition d), see item 4.

** If in problem a)

$$G(t) = \prod_{k=1}^{M}(t-\alpha_k)^{m_k}\prod_{k=1}^{N}(t-\beta_k)^{-n_k}G_0(t);$$

($m_k, n_k$ are positive integers, $G_0(t) \in S$, $G_0(t) \neq 0$), then we obtain the exceptional case of the Riemann problem, requiring special consideration. The proposed scheme, however, remains valid.

$\alpha(t)$ maps $L$ onto itself one-to-one with preservation of the direction of traversal in problem b) and reversal of it in problem c).

Together with problems (1), let us consider in the space $S$ the auxiliary problems

$$\begin{aligned}
&\mathrm{a}^*) \quad \varphi^+(t) = [G(t)]^{-1}\varphi^-(t) + \varphi(t), \quad t \in L;\\
&\mathrm{b}^*) \quad \varphi^+[\alpha(t)]\alpha'(t) = [G(t)]^{-1}\varphi^-(t) + \varphi(t), \quad t \in L;\\
&\mathrm{c}^*) \quad \varphi^+(t) = [G(t)]^{+1}\alpha'(t)\varphi^+(t) + \varphi(t), \quad t \in L;\\
&\mathrm{d}_1^*) \quad \mathrm{Re}\left[G(t)\frac{dt}{d\theta}\varphi^+(t)\right] = \varphi(t),\\
&\qquad\qquad t = e^{i\theta}\\
&\mathrm{d}_2^*) \quad \mathrm{Re}\left[G(t)i\frac{dt}{d\theta}\psi^+(t)\right] = \varphi(t).
\end{aligned} \tag{2}$$

Each of the problems $\mathrm{a}^*, \mathrm{b}^*, \mathrm{c}^*$ is called the adjoint with respect to the corresponding problem (1). With problem d) two problems $\mathrm{d}_1^*$ and $\mathrm{d}_2^*$ are adjoint.

We shall assume that everywhere in (2) $\varphi(t) \in S$, and in the problems $\mathrm{d}_1^*$ and $\mathrm{d}_2^*$, $\operatorname{Im}\varphi(t) = 0$.

In solving each of the problems (2), three different cases may occur: 1) the problem is solvable uniquely for any right-hand side $\varphi(t) \in S$; 2) the problem is solvable if and only if $\varphi(t)$ is orthogonal to a certain number of linearly independent functions; in this case the solution will be unique; 3) the problem is solvable for any right-hand side; the solution depends linearly on a certain number of arbitrary constants.

In cases 1) and 2), the solution will be represented in the form of a Cauchy-type integral whose density contains the multiplier function $\varphi(t)$. Using this fact, one can show that the solution belongs to the space $S$ and depends continuously on $\varphi(t)$. A particular solution possessing these properties can also be singled out in case 3).

3. We shall solve the Riemann problem a), assuming that $\chi = \operatorname{ind} G(t) = 0$. Let us express the functionals $(f^+, \varphi)$ and $(f^-, \varphi)$ through $G(t)$ and $(g, \varphi)$, using the fact that any fundamental function $\varphi(t)$ is uniquely representable in the form $\varphi(t) = \varphi^+(t) - [G(t)]^{-1}\varphi^-(t)$, since the problem a*), adjoint to a), for $\chi = 0$, is solvable uniquely. We have

   $$(f^+, \varphi(t)) = (f^+, \varphi^+(t) - [G(t)]^{-1}\varphi^-(t)) = -(f^+, [G(t)]^{-1}\varphi^-(t)) = -(G(t)f^- + g, [G(t)]^{-1}\varphi^-(t)) = -(g, [G$$

   Arguing analogously, we obtain

   $$(f^-, \varphi(t)) = -(g, [\varphi(t)/G(t)]^+),$$

   where $[\varphi(t)/G(t)]^+ = \psi^+(t)$ is the solution of the problem

   $$\psi^+(t) = [G(t)]^{-1}(\psi^-(t) + \varphi(t)).$$

   It is easy to verify that the constructed functionals satisfy the boundary condition a). Their continuity follows from the fact that $\varphi^{\pm}(t) \to 0$ when $\varphi(t) \to 0$, and from the continuity of the functional defining the generalized function $g$.

In an analogous way one can obtain the solution of problem b) for $\chi = 0$ and of problem c) for $\chi = -1$. For the Hilbert problem the cases $\chi = 0$ and $\chi < 0$ are essentially the same, and we shall carry out the corresponding investigation in the next item.

4. Let $\chi < 0$. Arguing as above, we consider together with a) the problem a*). Since the index of a*), equal to $-\chi$, will be positive, a linear combination with arbitrary coefficients of the solutions of the corresponding homogeneous problem will enter into its solution. From the relation

   $$(f^+, \varphi(t)) = (g, \varphi^+(t) - \varphi(t))$$

   it is seen that the functional $(f^+, \varphi(t))$ will be defined and continuous if and only if the conditions

   $$(g, \varphi_k^+(t)) = 0, \quad k = 1, 2, \dots, |\chi|,$$

> are satisfied, where $\varphi_k(t)$ are the solutions of the adjoint homogeneous problem
> $$\varphi^+(t) = [G(t)]^{-1}\varphi^-(t).$$
> These same conditions ensure the definiteness and continuity of the functional $(f^-, \varphi(t))$. Thus, they will be the solvability conditions for problem a). Analogous arguments are applicable also to problem b) for $\chi < 0$ and to problem c) for $\chi < -1$.

To investigate the Hilbert boundary-value problem d) in the space of generalized functions, let us rewrite its boundary condition in the form $f^+ = (g + i\rho)G(t)$, where $\rho$ is the required real generalized function, i.e., one such that $\operatorname{Im}\left(\rho, \varphi(t)/\dfrac{dt}{d\theta}\right) = 0$ for all real test functions $\varphi(t)$. If $\operatorname{ind} G(t) = \varkappa \le 0$, then each of the problems $\Gamma_1^*$ and $\Gamma_2^*$ has $1 - 2\varkappa$ linearly independent solutions. It can be shown that

$$(f^+, \varphi(t)) = (g, G(t)[2\varphi(t) - \varphi^+(t) - \psi^+(t)]). \tag{3}$$

This functional is continuous if, for any solution $\varphi_k^+(t)$, $k = 1, 2, \dots, 1-2\varkappa$, of the homogeneous problem $\operatorname{Re}\left[G(t)\dfrac{dt}{d\theta}\varphi^+(t)\right] = 0$, the conditions $(g, G(t)\varphi_k^+(t)) = 0$ are satisfied. When these conditions are fulfilled, formula (3) gives a solution of problem d).

5. The adjoint problem may turn out to be solvable only when conditions of the form

$$\int_L \varphi(t)h_j(t)\,dt = 0, \qquad j = 1, 2, \dots, N, \tag{4}$$

are satisfied, where $h_j(t)$ are certain linearly independent test functions. Such a situation, for example, occurs for $\varkappa > 0$ in problems a*) and b*). Thus, in this case we can define the required functionals only on the subspace $S_h$ of the test-function space $S$, consisting of test functions $\varphi(t)$ satisfying conditions (4).

To each test function $\Phi(t)$ let us assign a function $\varphi(t) \in S_h$ by setting

$$\varphi(t) = \Phi(t) - \sum_{k=1}^{N} \Phi_k(t) \int_L \Phi(t)h_k(t)\,dt. \tag{5}$$

Here the test functions $\Phi_k(t)$ are chosen so that $\int_L \Phi_k(t)h_m(t)\,dt = \delta_{km}$, $k, m = 1, 2, \dots, N$ ($\delta_{km} = 0$ for $k \ne m$, $\delta_{kk} = 1$). The construction of such functions is easy to carry out by slightly modifying the method described in (7), pp. 567-569.

**Theorem.** *A functional $(f,\varphi)$, defined and continuous on the subspace $S_h$, extends, with preservation of continuity, to the whole space $S$ by the formula*

$$(f,\Phi)=(f,\varphi)+\int_L\sum_{k=1}^{N}C_k h_k(t)\Phi(t)\,dt, \tag{6}$$

where $C_k$ are arbitrary constants; $\varphi(t)$ is determined from $\Phi(t)$ by relation (5). The extension possessing the indicated properties is unique up to the choice of the constants $C_k$.

The proof is carried out by direct verification of all assertions of the theorem.

The theorem makes it possible to complete the solution in the case $\varkappa>0$. For example, for the Riemann problem $h_k(t)=X^+(t)t^{k-1}$, $k=1,2,\dots,\varkappa$. Thus we arrive at the expression

$$(f^+,\Phi(t))=(f^+,\varphi(t))+\int_L\sum_{k=1}^{\varkappa}C_k t^{k-1}X^+(t)\Phi(t)\,dt=$$

$$=-\left(g,\frac{\varphi^-(t)}{G(t)}\right)+\int_L\sum_{k=1}^{\varkappa}C_k t^{k-1}X^+(t)\Phi(t)\,dt. \tag{7}$$

Taking into account that $\varphi^\pm(t)$ is a solution of the problem

$$\varphi^+(t)-\frac{\varphi^-(t)}{G(t)}=\Phi(t)-\sum_{k=1}^{\varkappa}\Phi_k(t)(h_k(t),\Phi(t)),$$

and using the representation of $G(t)$ in terms of the limiting values of the canonical function $X(z)$ of problem $a^*$, we give formula (7) the form

$$(f^+,\Phi(t))=(X^+(t)\{(g/X^+(t))+P_{\varkappa-1}(t)\},\Phi(t)), \tag{8}$$

where $P_{\varkappa-1}(z)$ is a polynomial of degree $\varkappa-1$ with arbitrary coefficients. This form of the solution is completely analogous to that which it has in the classical case.

6. In conclusion, let us consider the linear equation

$$Kf=g, \tag{9}$$

regarding $g$ as a generalized function on the basic space $S$. We define the operator $K$ by means of the adjoint operator $K^*$ through the relation $(Kf,\varphi)=(f,K^*\varphi)$, which must reduce to an identity if $f\in S$. Suppose the equation

$K^*\psi = \varphi$, $\varphi \in S$, $\psi \in S$, is solvable under the fulfillment of $\alpha$ ($\alpha \geqslant 0$) conditions of the form

$$\int_L \varphi(t) h_j(t)\, dt = 0, \quad j = 1, 2, \dots, \alpha,$$

where $h_j(t)$ are certain linearly independent basic functions, and moreover the solution has the form

$$\psi(t) = \psi_0(t) + \sum_{k=1}^{\beta} A_k \psi_k(t),$$

where $\psi_0(t) \to 0$ if $\varphi(t) \to 0$; $A_k$ are arbitrary constants; $\beta \geqslant 0$. Under these conditions the functional $(f, \varphi)$ on the subspace $S_h$ is defined as follows:

$$(f, \varphi) = (f, K^*\psi) = (Kf, \psi) = (g, \psi) = \left( g, \psi_0 + \sum_{k=1}^{\beta} A_k \psi_k(t) \right).$$

For the continuity of the functional $(f, \varphi)$ on $S_h$ it is necessary and sufficient that the conditions

$$(g, \psi_k(t)) = 0, \quad k = 1, 2, \dots, \beta$$

be satisfied. According to the theorem of item 5, the functional on the whole space $S$ is obtained by the formula

$$(f, \Phi(t)) = (f, \varphi(t)) + \left( \sum_{k=1}^{\alpha} C_{kh} k(t), \Phi(t) \right) = (g, K^{*-1}\varphi) + \left( \sum_{k=1}^{\alpha} C_{kh} k(t), \Phi(t) \right),$$

where $C_k$ are arbitrary constants. Thus, the functions $h_k(t)$ will be solutions of the homogeneous equation $Kf = 0$.

The indicated scheme can be applied to the study, in the class of generalized functions, of singular integral equations with Cauchy kernel and of paired integral equations of convolution type.

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*Note: Figure translations are in progress. See original paper for figures.*