ON DUALITY PRINCIPLES

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Abstract

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ON DUALITY PRINCIPLES

FOR THE PROBLEM OF PERIODIC SOLUTIONS

OF DIFFERENTIAL EQUATIONS OF HIGHER ORDERS

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As is known, the question of the solvability of a number of complicated problems reduces to finding fixed points of the corresponding transformations A. In this connection, for one and the same problem it may be convenient to pass to several transformations. Suppose that a certain problem has been reduced to the question of fixed points of two operators U and A, acting, generally speaking, in different spaces. The natural question arises of formulas relating the corresponding topological characteristics of the operators U and A. Such formulas, following M. A. Krasnosel' skii, will be called duality theorems. A number of duality theorems for the problem of periodic solutions of a system of first-order differential equations was established by M. A. Krasnosel' skii and V. V. Strygin (^{5,7}). As was shown by M. A. Krasnosel' skii (^{6}), such duality theorems lead to substantially new conditions for the existence of periodic solutions for equations with deviating arguments and for more general classes of evolutionary problems.

In the present paper, duality theorems are presented for the problem of periodic solutions of systems of differential equations of higher order. The theorems proposed are new also for the case of systems of first-order equations; the results of papers (^{5,7}) fit into them.

We note that the examples of integral operators considered in Sec. 3 were also considered in (^{5}), and also that one particular duality principle for equations of second order was indicated by M. A. Krasnosel' skii.

The results of the present work can be applied to the investigation of periodic solutions of higher-order equations with deviating arguments. Here the scheme of reasoning proposed in (^{6}) is applicable.

1. Consider the vector differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f(t, y, y', \dots, y^{(n-1)}). \tag{1}$$

Assume that the matrices $a_i(t)$ and the vector-function $f(t, y, y', \dots, y^{(n-1)})$ are continuous and possess the property of ω-periodicity in t:

$$\begin{gathered} a_i(t+\omega) \equiv a_i(t) \quad (i = 1, 2, \dots, n); \\ f(t+\omega, y, y', \dots, y^{(n-1)}) \equiv f(t, y, y', \dots, y^{(n-1)}). \end{gathered} \tag{2}$$

Assume also that all solutions of equation (1) can be continued to an interval of length ω. Let $y(t)$ be a k-dimensional vector-function. By the initial condition of $y(t)$ we shall mean the kn-dimensional vector z_y, the first k coordinates of which coincide with the coordinates of the vector $y(0)$, the next k with the coordinates of the vector $y'(0)$, and so on; the last k coordinates of the vector z_y coincide with the coordinates of the vector $y^{(n-1)}(0)$. In what follows we shall assume that to each initial condition z from the kn-dimensional space R there corresponds one and only one solution $y(t)$ of differential equation (1).

The problem of the existence of periodic solutions of equation (1) can be reduced to the problem of fixed points of various operators.

Choose an arbitrary element $z \in R$. Let $q(t, z)$ denote the solution of equation (1) with initial condition z. In view of (2), the vector-function $q(t+\omega, z)$ is also a solution of equation (1); consequently, $q(t+\omega, z) = q(t, z_1)$, $z_1 \in R$. The operator $Uz = z_1$ will be called the shift operator along the trajectories of the differential equation (1).

It is known that the fixed points of the operator U, acting in the space R, completely describe the set of periodic solutions of equation (1), namely, in order that a solution $q(t, z^*)$ of equation (1) be ω-periodic, it is necessary and sufficient that the equality $Uz^* = z^*$ hold.

Let us introduce the homogeneous equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = 0. \tag{3}$$

The solution of equation (3) with initial condition z will be denoted by $p(t, z)$. The shift operator V along the trajectories of equation (3), defined analogously to the operator U, is a linear operator.

The solutions of equation (3) form a subspace E of the space of C^{n-1} vector-functions, $n-1$ times continuously differentiable on the interval $[0, \omega]$. Between the spaces E and R there exists a natural isomorphism B, given by the equality

$$Bp(t, z) = z \quad (z \in R). \tag{4}$$

The following assertions make it possible to construct operators A, acting in the space C^{n-1}, whose fixed points describe the set of periodic solutions of equation (1).

Theorem 1. Let a completely continuous operator A, acting in the space C^{n-1}, satisfy the following conditions:

1) $Ax(t)$ is a solution of the differential equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = f[t, x(t), x'(t), \dots, x^{(n-1)}(t)]. \tag{5}$$

2) If $x(t) = q(t, z)$, then

$$Ax(t) = x(t) - p[t, C(z - Uz)], \tag{6}$$

where C is a nonsingular matrix.

Then the set of fixed points of the operator A coincides with the set of ω-periodic solutions of equation (1).

Theorem 2. Let 1 not be an eigenvalue of the operator V. Let a completely continuous operator A satisfy condition 1) of Theorem 1 and the condition

2′) If $x(t) = q(t, z)$, then

$$Ax(t + \omega) = Ax(t). \tag{7}$$

Then the set of fixed points of the operator A coincides with the set of ω-periodic solutions of equation (1).

2. Let $G \subset R$ and $\Omega \subset C^{n-1}$ be bounded domains such that the set of fixed points of the operator U lying in G coincides with the set of initial values of the ω-periodic solutions of equation (1) lying in Ω, and suppose that on the boundary Γ of the domain Ω there are no ω-periodic solutions of equation (1), while on the boundary Π of the domain G there are no fixed points of the operator U.

The question arises as to how the rotations of the vector fields

$$\Phi z = z - Uz, \tag{8}$$

$$\Psi x = x - Ax \tag{9}$$

are related.

on the boundaries Π and Γ, respectively. It turns out that these rotations are equal in absolute value; more precisely, the following assertions hold.

Theorem 3. *Let the completely continuous operator A satisfy the conditions of Theorem 1.*

Then the rotation γ of the continuous vector field (8) on the boundary Π of the domain G and the rotation γ_1 of the completely continuous vector field (9) on the boundary Γ of the domain Ω are related by the equality

$$\gamma_1 = (-1)^{\alpha}\gamma,$$

where α is the sum of the multiplicities of the negative eigenvalues of the matrix C.

A consequence of Theorem 3 is

Theorem 4. *Suppose that the conditions of Theorem 2 are satisfied.*

Then the rotations γ and γ_1 are related by the equality

$$\gamma_1 = (-1)^{\beta}\gamma,$$

where β is the sum of the multiplicities of the eigenvalues of the operator V greater than 1.

For the proof of Theorem 3 an auxiliary domain $\Omega_1 \subset C^{n-1}$ is constructed,

$$\Omega_1 = \{x(t);\, x(t) \in C^{n-1},\, z_x \in G,\, \|x(t)\| < r\},$$

where z_x is the initial value of the function $x(t)$, and $r > 0$ is a number such that the functions $q(t,z)$ and $p(t,z)$, for $z \in R$,

$$\|z\| \le \sup_{z\in G}(\|C\|\,\|z - Uz\| + \|z\|)$$

lie in the open ball of the space C^{n-1} of radius $r/2$. It is shown that the rotation of the field Ψx on the boundary Γ_1 of the domain Ω_1 is equal to γ_1. By means of two homotopic transitions it is proved that γ_1 is equal to the rotation on Γ_1 of the vector field $x - \chi x$, where

$$\chi x(t) = p(t, z_x) - p[t, C(z_x - Uz_x)].$$

The rotation of the field $x - \chi x$ is equal to the rotation of the field

$$Px(t) = p[t, C(z_x - Uz_x)]$$

on the boundary of the domain $\Omega_1 \cap E$ of the space E. To prove the theorem it remains to apply the isomorphism B of the spaces E and R, defined in § 1.

To prove Theorem 4 it is shown that from its conditions there follow the conditions of Theorem 3 for $C = (1 - V)^{-1}$.

3. Let us give several very simple examples of operators satisfying the conditions of Theorems 1 and 2.

1) Let the matrices $U_i(t)$ $(i = 0, 1, \dots, n-1)$ be solutions of equation (3) and satisfy the conditions

$$U_i^{(j)}(0) = \delta_{ij} I \qquad (0 \le i, j \le n-1).$$

Then the solution of equation (5) with the initial conditions

$$y^{(i)}(0) = x^{(i)}(0)$$

can be written in the form

$$y(t) = \sum_{i=0}^{n-1} U_i(t) x^{(i)}(0) + \int_0^t J(t,s) f[s, x(s), x'(s), \dots, x^{(n-1)}(s)]\, ds,$$

where $J(t,s)$ is the Cauchy matrix.

The operator

$$Ax(t) = \sum_{i=0}^{n-1} U_i(t) x^{(i)}(\omega) + \int_0^t J(t,s) f[s, x(s), x'(s), \dots, x^{(n-1)}(s)]\, ds$$

satisfies the conditions of Theorem 1 for $C = I$.

2) Let $n = 1,\ a_1(t) \equiv 0$. Then the operator

$$Ax(t) = \frac{1}{\omega} \int_0^\omega \{x(s) + s f[s, x(s)]\}\, ds + \int_0^t f[s, x(s)]\, ds$$

satisfies the conditions of Theorem 1 for $C = 1$.

3) Let 1 not be an eigenvalue of the operator V. Then the operator

$$Ax(t) = \int_0^\omega G(t,s) f[s, x(s), x'(s), \dots, x^{(n-1)}(s)]\, ds,$$

where $G(t,s)$ is the Green' s function of the boundary-value problem

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y = g(t),$$

$$y^{(i)}(0) - y^{(i)}(\omega) = 0 \quad (i = 0, 1, \dots, n-1),$$

satisfies the conditions of Theorem 2.

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Note: Figure translations are in progress. See original paper for figures.

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