ANALYTIC CONDITIONS FOR THE POSITIVITY OF A REAL FUNCTION

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Abstract

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ANALYTIC CONDITIONS FOR THE POSITIVITY OF A REAL FUNCTION

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The solution of a number of very important problems in the theory of automatic control, circuit theory, etc., reduces to finding conditions for the positivity of real functions. We shall give some of them.

1. Sufficient conditions for absolute stability in the sector $[0, K]$ ([1]) for the case in which the characteristic of the nonlinear element has the form

$$0 \leq \varphi(\sigma)/\sigma \leq K, \tag{1}$$

where K is a finite positive number or infinity, reduce to determining condition (2)

$$\mathrm{Re}(1 + ig\omega)W(i\omega) + \frac{1}{K} > 0, \tag{2}$$

where g is a finite real number; $W(i\omega)$ is the amplitude-phase characteristic of the open linear part of the system; here the system is opened at the place where the element with the nonlinear characteristic is located.

In works ([1,2]) it was shown that sufficient conditions for absolute stability of the system in the sector $[0, K]$ will be satisfied if, in the plane $W^*(i\omega) = i\omega W$, one can choose a straight line passing through the point of the real axis with abscissa $-\frac{1}{K}$ in such a way that the modified characteristic $W^*(i\omega)$ lies entirely strictly to the right of this straight line and so that, in addition, in special cases ([2]) limiting stability of the system holds.

It is of substantial interest to find analytic conditions under which ([2]) is satisfied.

2. As is known, the conditions of physical realizability of a circuit obtained as a result of synthesis reduce to the necessity of determining the positivity of a real function. However, up to the present time there have been no analytic conditions, expressed in terms of the circuit parameters or the coefficients of an equation, under which a positive real function is realized.

The examples indicated represent only some of a large number of problems whose solution reduces to determining the positivity of a real function.

As is known, a function $z(p)$ is called positive if its real part is positive for positive real parts of p:

$$\operatorname{Re} z(p) \geq 0 \quad \text{for} \ \operatorname{Re} p \geq 0. \tag{3}$$

If the function takes real values for real values of the argument, then it is called a real positive function.

From (3) and (2) we see that for real values of ω the condition of the theorem of P. M. Popov ([1]) reduces to finding conditions for the positivity of a real function.

From the properties of real positive functions it follows that the conditions for their realizability will be found if the conditions for stability of the equation

$$Q(p) + KR(p) = 0 \tag{4}$$

are found for all values of K.

The problem may be formulated as follows. Find the conditions under which the region of stability in the plane of the complex parameter $\overline{K}$ will contain the entire positive real axis.

First of all it is clear that the polynomials $Q(p)$ and $R(p)$ must be Hurwitz polynomials, which follows from the stability requirements of (4), respectively for $K = 0$ and $K = \infty$.* In order that there exist a segment of the real axis belonging to the stability region and beginning at infinity, it is necessary and sufficient that the structure of the system corresponding to equation (4) belong to the class of stable systems as $K \to \infty$ (4), and that the necessary and sufficient conditions formulated in (4) be satisfied. Suppose that there exists a segment on the real axis K of the plane $\overline{K}$, beginning at infinity and belonging to the stability region.

Relying on the theorem on the continuous dependence of the roots on the coefficients of an equation, we may assert that, in addition to the segment of the stability region beginning at infinity, there exists a segment of the real axis K of the plane $\overline{K}$, beginning at the origin, which also belongs to the stability region. Under these conditions the following proposition may be formulated. If $Q(p)$ and $R(p)$ are Hurwitz polynomials and there exists a segment of the real axis K of the plane $\overline{K}$, beginning at infinity and belonging to the stability region, then the entire real axis K of the plane $\overline{K}$ will belong to the stability region, provided that the imaginary part of the expression $\overline{K}$ has no positive real roots. The validity of this proposition follows directly from the properties of the D-partition curve. Thus, it remains to find the conditions under which, in structures stable for an unbounded increase of K, the D-partition curve never intersects the real axis K of the plane $\overline{K}$.

From (4), the equation of the D-partition curve is written in the form

$$K = -\frac{Q(j\omega)}{R(j\omega)} = -\frac{Q_1(\omega) + jQ_2(\omega)}{R_1(\omega) + jR_2(\omega)}. \tag{5}$$

The imaginary part of equation (5) is written in the form

$$\frac{Q_1(\omega)R_2(\omega) - R_1(\omega)Q_2(\omega)}{R_1^2(\omega) + R_2^2(\omega)}. \tag{6}$$

Consequently, in order that the D-partition curve not intersect the K axis, it is necessary and sufficient that

$$Q_1(\omega)R_2(\omega) - R_1(\omega)Q_2(\omega) = 0 \tag{7}$$

have no real roots.

In the general case, if the zero root is omitted, equation (7) may be written in the following form:

$$\sum_{i=0}^{n} a_i \omega^{2i} = 0. \tag{8}$$

It follows from what has been said that the conditions of reality and positivity of the function $KR(p)/Q(p)$ reduce to determining the Hurwitz property of the polynomials $Q(p)$ and $R(p)$, and to the absence of real roots in the polynomial containing only even powers of ω and representing the imaginary part of the expression of the D-partition curve with respect to K.

Another approach is also legitimate. Namely, one may require that the real part of $KR(j\omega)/Q(j\omega)$ for all $\omega > 0$ be greater than zero; in

* $Q(p)$ may have one zero root.

under these conditions the function will be real and positive, and in this case the numerator of the real part of the function, which is a polynomial with even powers, must not have real roots.

Consider the polynomial (8); we have

$$\sum_{i=0}^{n} a_i \omega^{2i}. \tag{9}$$

The degree of the polynomial (9) is $2n$.

Consider the auxiliary polynomial of degree $n + 1$

$$f(x) = b_{n+1}x^{n+1} + b_n x^n + ... + b_0. \tag{10}$$

We require that the roots of (10) be real and distinct. For this, the coefficients of (10) must satisfy the system of inequalities found in (5).

If the roots of (10) are real and distinct, then the roots of $f'(x)$ will also be real and distinct and will interlace with the roots of (10), and the roots of $f''(x)$ will likewise be real and will interlace with the roots of $f'(x)$.

The following theorem is known[6]:

If the roots of the functions $f(x)$ and $\varphi(x)$ are real, distinct, and interlacing, then the equation

$$f'(x)\varphi(x) - f(x)\varphi'(x) = 0 \tag{11}$$

has no real roots.

If we set $\varphi(x) = f'(x)$, then the equation

$$[f'(x)]^2 - f(x)f''(x) = 0 \tag{12}$$

has no real roots.

On the basis of (10) it is clear that the degree of (12) will be $2n$.

Equate the polynomials (9) and (12)

$$\sum a_i \omega^{2i} = [f'(x)]^2 - f(x)f''(x). \tag{13}$$

From (13) we obtain a system of recurrent relations for determining the coefficients of the polynomial $f(x)$ in terms of the coefficients of the polynomial (9).

Denote the newly obtained coefficients of the polynomial $f(x)$ by $A_{n+1}, A_n, ..., A_0$ (some of these coefficients may be equal to zero) and form the determinant expressing the conditions for the reality of the roots (5) of this polynomial.

We have

$$\begin{vmatrix} (n+1)A_{n+1}, & A_{n+1}, & 0, & ... & ..., & 0 \\ nA_n, & A_n, & (n+1)A_{n+1}, & A_{n+1}, & 0, ..., 0 & \\ ... & ... & ... & ... & ... & ... \\ 0, & ... & 0, & (n+1)A_{n+1}, & A_{n+1} & \end{vmatrix}. \tag{14}$$

Thus, in order that the polynomial (9) have no real zeros (the condition for positivity of the real function), the principal minors of the determinant (14), composed of the coefficients of the auxiliary polynomial (10) expressed in terms of the coefficients of the polynomial (9), must be greater than zero.

Remark. Since theorem [6] has no converse, the conditions obtained are sufficient. However, as calculations have shown for a number of important cases, these conditions are also necessary. Moreover, for each specific case one can also check the necessary conditions, since for real x (12) will be greater than zero.

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Note: Figure translations are in progress. See original paper for figures.