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Abstract

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MATHEMATICS

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POSITIVE AND NEGATIVE NUMBERINGS

A numbering β of a collection of certain objects \mathfrak{B} is a single-valued mapping of the set N of all natural numbers onto \mathfrak{B} . The numbering equivalence θ_β is the binary relation on N defined by the formula $x\theta_\beta y \iff \beta x = \beta y$. If the numbering β is isomorphic (for the terminology see ⁽¹⁾) to some computable numbering of a suitable family of recursively enumerable sets (r.e.s.), then there exists a general recursive function (g.r.f.) $F(x, y, u, v)$ for which

$$\beta x = \beta y \iff \forall u \exists v (F(x, y, u, v) = 0).$$

Thus, the numbering equivalences of computable numberings of families of r.e.s. have class $\forall\exists$ in Kleene's classification. Below we consider in greater detail the properties of numberings of class I, for which the equivalence θ has either the form

$$\beta x = \beta y \iff \exists u (F(x, y, u) = 0),$$

or the form

$$\beta x = \beta y \iff \forall u (F(x, y, u) = 0),$$

where $F(x, y, u)$ is a suitable g.r.f. In the first case, the set of pairs $\langle x, y \rangle$ for which $\beta x = \beta y$ is recursively enumerable, and the numbering β is called **positive**. In the second case, the numbering β is called **negative**. Under a negative numbering, the set of pairs $\langle x, y \rangle$ for which $\beta x \neq \beta y$ turns out to be recursively enumerable. A numbering β is **decidable** if it is simultaneously positive and negative.

1. Let the collection \mathfrak{A} have some numbering α . A numbering β of a subcollection $\mathfrak{B} \subseteq \mathfrak{A}$ is called **α -computable** if $\beta n = \alpha f(n)$ for a suitable g.r.f. f . A numbering β of the collection \mathfrak{B} is called **uniformizable** if there exists a one-to-one β -computable numbering of \mathfrak{B} . A subcollection $\mathfrak{B} \subseteq \mathfrak{A}$ is called **α -uniform** if there exists an α -computable one-to-one numbering of \mathfrak{B} . A family $\mathfrak{B} \subseteq \mathfrak{A}$ is called **α -isolated** if α -computable numberings of \mathfrak{B} exist and all of them are equivalent to one another (i.e., computable relative to one another). If the basic collection is the collection of all r.e.s. and α is the usual Post (= Gödel) numbering π , then instead of π -computable, π -uniform, π -isolated we shall say computable, uniform, and so on.

Theorem 1. *If the basic numbering α of the collection of objects \mathfrak{A} is positive (negative), then every α -computable numbering β of an arbitrary subcollection $\mathfrak{B} \subseteq \mathfrak{A}$ is also positive (negative). If α is positive and the subcollection \mathfrak{B} is α -computable (i.e., possesses at least one α -computable numbering), then \mathfrak{B} is α -isolated.*

The first assertion follows directly from the definitions. We prove the second. By assumption, the set θ of pairs $\langle x, y \rangle$ for which $\alpha x = \alpha y$ can be arranged in a recursive sequence

$$\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle, \dots$$

Let β, γ be arbitrary α -computable numberings of \mathfrak{B} , and let

$$\beta n = \alpha f(n), \quad \gamma n = \alpha g(n),$$

where f, g are suitable g.r.f. For a given n , in the sequence

$$\langle x_0, y_0 \rangle, \langle x_1, y_1 \rangle, \dots$$

we look for a pair of the form $\langle f(n), g(x) \rangle$ and put $x = h(n)$. Then $\beta n = \gamma h(n)$.

Corollary. *Uniformizable positive numberings are decidable.*

Indeed, let α be positive, and let β be an α -computable one-to-one numbering of the collection $\mathfrak{B} = \mathfrak{A}$. According to the preceding theorem, α and β are equivalent. But β is decidable, and therefore α is decidable.

§ 2. A family of r.e. sets \mathfrak{B} will be called **finitely separable** if there exists a **strongly** enumerable sequence of finite sets a_0, a_1, \dots such that each of the sets of the family \mathfrak{B} contains at least one of the indicated finite sets, and each of these finite sets is contained in no more than one of the sets of the family \mathfrak{B} . For example, if the family \mathfrak{B} is a partition (i.e., the sets \mathfrak{B} are nonempty, pairwise disjoint, and their union gives N), then \mathfrak{B} is finitely separated by the sequence $\{0\}, \{1\}, \dots$

Theorem 2. *Every positive numbering α of an arbitrary collection of objects is isomorphic to a computable numbering of the corresponding partition N/θ_α . Every finitely separable family of r.e. sets \mathfrak{B} that has a computable numbering β is an isolated positive family.*

Let

$$[a] = (x \mid \alpha x = \alpha a).$$

By assumption, for the occurring partial recursive function $F(a, x, y)$ we have

$$\alpha x = \alpha a \iff \exists y (F(a, x, y) = 0).$$

Consequently, $[a]$ is the domain of definition of the partial recursive function

$$\Phi(a, x) = \mu_y(F(a, x, y) = 0),$$

and therefore the mapping $\alpha_1 : a \rightarrow [a]$ is a computable numbering of the partition N/θ_α . Since

$$aa = ab \iff \alpha_1 a = \alpha_1 b,$$

the numberings α and α_1 are isomorphic.

To prove the second assertion, consider a strongly enumerable sequence of finite sets a_0, a_1, \dots finitely separating \mathfrak{B} . Let β be an arbitrary computable numbering of \mathfrak{B} . Taking an arbitrary n and successively computing the elements of the sets $nn, \beta i, a_i$ ($i, j = 0, 1, \dots$), we look for such i, j that

$$a_i \subseteq nn, \quad a_i \subseteq \beta j.$$

Denote by $g(n)$ the first j encountered in this process and satisfying the indicated conditions. It is clear that $g(n)$ is a partial recursive function and that

$$nn \in \mathfrak{B} \Rightarrow g(n) \text{ is defined and } nn = \beta g(n).$$

This means that an arbitrary computable numbering β of the family \mathfrak{B} is subnormal⁽¹⁾. But all subnormal numberings are equivalent⁽¹⁾, and therefore the family is isolated. It remains to show that the numbering β is positive. Computing successively the elements of the sets $\beta j, a_i$ ($i, j = 0, 1, \dots$) and marking those pairs $\langle a, b \rangle$ for which, at some moment of time, it will be

$$a_i \subseteq \beta a \quad \text{and} \quad a_i \subseteq \beta b,$$

we enumerate all pairs for which

$$\beta a = \beta b,$$

as required.

It follows from Theorem 2 that *if some finitely separable family admits a computable unsolvable numbering, then it is not uniform*. The simplest example of such a family is constructed as follows. Let A be a nonrecursive recursively enumerable set of numbers:

$$A = \{a_0, a_1, a_2, \dots\}.$$

We construct the sets A_0, A_1, \dots , putting into each A_i , at the n -th step, the number i , while into the sets A_{2a_n} and A_{2a_n+1} we put the numbers $2a_n$ and $2a_n + 1$. The numbering

$$\alpha : i \rightarrow A_i$$

is a computable positive numbering of the partition $\{A_0, A_1, \dots\}$. This numbering is **unsolvable**, since

$$A_{2i} = A_{2i+1} \iff i \in A.$$

Putting

$$\beta n = A_0 \cup A_1 \cup \dots \cup A_n,$$

we obtain a computable numbering of a **nondecreasing** sequence of finite sets:

$$\beta 0 \subseteq \beta 1 \subseteq \dots, \quad \bigcup \beta n = N.$$

The numbering β is isomorphic to α and therefore is positive and unsolvable. According to the corollary to Theorem 1, the numbering β is not uniformizable. At the same time, from the arguments of Friedberg ⁽²⁾ it follows (see ⁽³⁾) that every computable family of r.e. sets containing a strongly enumerable sequence of increasing finite sets admits a one-one computable numbering. Therefore the family βN certainly has a one-one computable numbering and, consequently, may serve as an example of a uniform family admitting a nonuniformizable positive computable numbering.

§ 3. Let β be a computable numbering of some family of partial recursive functions of one variable. This means that, for a suitable partial recursive function $U(n, x)$, we have the identity

$$(\beta n)(x) = U(n, x).$$

Computing successively $U(0, 0)$,

$U(0, 1), U(1, 0), \dots$ and marking those pairs $\langle a, b \rangle$ for which, for some x , $U(a, x) \neq U(b, x)$, we enumerate in general all pairs $\langle a, b \rangle$ for which $\beta a \neq \beta b$. Therefore all computable numberings of families of general recursive functions are negative. The usual numbering of the family of all primitive recursive functions ⁽³⁾ may serve as a typical example of an undecidable negative numbering.

Theorem 3. *Every negative numbering α of an arbitrary family of objects is isomorphic to a suitable computable numbering of a suitable family of general recursive functions taking only the values 0, 1.*

By hypothesis, the aggregate $-\theta$ of all pairs $\langle a, b \rangle$ satisfying the relation $aa \neq ab$ can be arranged in a recursive sequence

$$-\theta = \{\langle a_0, b_0 \rangle, \langle a_1, b_1 \rangle, \dots\}.$$

We construct a function $U(n, x)$ and auxiliary sets S_{i0}^m, S_{i1}^m by means of the following algorithm. For an arbitrary m put

$$U(a_m, m) = 0, \quad U(b_m, m) = 1, \quad S_{00}^m = \{a_m\}, \quad S_{01}^m = \{b_m\}.$$

Suppose that, for some s , finite sets S_{s0}^m, S_{s1}^m have already been constructed so that

$$i \in S_{s0}^m \Rightarrow U(i, m) = 0$$

and

$$i \in S_{s1}^m \Rightarrow U(i, m) = 1,$$

and moreover, if $i \in S_{s0}^m, j \in S_{s1}^m$, then $\langle i, j \rangle \in -\theta$. Let x be the least natural number not contained in

$$S_{s0}^m \cup S_{s1}^m.$$

Take some pair $\langle a, b \rangle$, where $a \in S_{s0}^m, b \in S_{s1}^m$. Since $aa \neq ab$, we have $aa \neq ax$ or $ab \neq ax$, and consequently, by running through the pairs in the sequence $-\theta$, we shall find in this sequence either a pair of the form $\langle a, x \rangle$, or a pair of the form $\langle b, x \rangle$. For each pair $\langle a, b \rangle$, $a \in S_{s0}^m, b \in S_{s1}^m$, select a pair $\langle u_{a,b}, x \rangle \in -\theta$, where $u_{a,b} = a, b$. Among the selected pairs there necessarily occur either all pairs of the form $\langle a, x \rangle$, $a \in S_{s0}^m$, or all pairs of the form $\langle b, x \rangle$, $b \in S_{s1}^m$. In the first case put

$$U(x, m) = 1, \quad S_{s+11}^m = S_{s1}^m \cup \{x\}, \quad S_{s+10}^m = S_{s0}^m,$$

and in the second

$$U(x, m) = 0, \quad S_{s+10}^m = S_{s0}^m \cup \{x\}, \quad S_{s+11}^m = S_{s1}^m.$$

The induction assumptions are thereby preserved and the function $U(x, m)$ is constructed. It is clear that

$$aa = ab \iff \forall x (U(a, x) = U(b, x)),$$

and therefore the computable numbering $n \mapsto U(n, x)$ is isomorphic to the numbering α .

Among the various numberings of a family, the precomplete ones (3) are usually of interest. However, for negative numberings the following holds.

Remark. *No negative numbering α of an aggregate \mathfrak{A} consisting of more than one object can be precomplete.*

Indeed, let a, b be distinct objects of \mathfrak{A} . The aggregate of all α -numbers of all objects distinct from a is recursively enumerable. Similarly, the aggregate of

all α -numbers of all objects distinct from b is recursively enumerable. In other words, the families $\mathfrak{A} - a$ and $\mathfrak{A} - b$ are α -completely enumerable. They are distinct from \mathfrak{A} and their union coincides with \mathfrak{A} ; but this, according to (1), cannot occur for precomplete numberings.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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