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Abstract

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THE THREE-CYLINDER THEOREM AND ITS APPLICATIONS

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This note considers a three-cylinder theorem for a parabolic equation of second order, analogous to Hadamard' s three-circle theorem for analytic functions of a complex variable (^{1}).

Denote by $\Pi_r^{[t_1,t_2]}$ the cylinder

$$(x_1^2+\dots+x_n^2)^{1/2}=|x|\le r,\qquad t_1\le t\le t_2,$$

and, for a function $f(x,t)$ continuous in the cylinder $\Pi_r^{[t_1,t_2]}$, put

$$M_f(r,t_1,t_2)=\max_{(x,t)\in\Pi_r^{[t_1,t_2]}}|f(x,t)|.$$

Suppose that in the cylinder $\Pi_R^{[0,T]}$, $R\le 1$, the equation

$$Hu\equiv\sum_{i,k=1}^{n}a_{ik}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_k}+\sum_{i=1}^{n}b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t}=0 \tag{1}$$

is defined such that: 1) $a_{ik}(x,t)$ are three times continuously differentiable with respect to x_i and continuously differentiable with respect to t; 2)

$$\sum_{i,k=1}^{n}a_{ik}(x,t)\xi_i\xi_k\ge\mu\sum_{i=1}^{n}\xi_i^2,\qquad \mu>0;$$

3) the coefficients $a_{ik}(x,t)$ and their derivatives up to the indicated order, as well as the coefficients $b_i(x,t)$ and $c(x,t)$, are bounded in modulus by the constant K.

Theorem 1 (on three cylinders). Let $u(x,t)$ be a solution of equation (1) satisfying the conditions

$$M_u(r_0,0,T)=\Delta;\qquad M_u(R,0,T)=M. \tag{2}$$

Then for any r, $r_0<r\le R/2C$, and any positive a, the inequality

$$\ln M_u(r,a,T-a)\le\ln M_u(r_0,0,T)\frac{\ln(Cr)}{\ln(Br_0)}+\ln M_u(R,0,T)\frac{\ln(Br_0/Cr)}{\ln(Br_0)}-\ln(Cr), \tag{3}$$

holds, where B, C are constants depending on the equation, a, and n.

Proof. According to the results of Li and Dž-huan (^{2}), there exists a transformation of coordinates $x = x(y)$ that maps $\Pi_R^{[0,T]}$ into a domain containing the cylinder $\Pi_s^{*[0,T]}$, $|y| = \rho \leq s,\ 0 \leq t \leq T$, the cylinder $\Pi_{\rho_0}^{*[0,T]}$ into a domain containing the cylinder $\Pi_{r_0}^{[0,T]}$, and transforms the operator H into an operator H^* of the form

$$H^* \equiv \frac{\partial^2}{\partial \rho^2} + \frac{n-1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2} M_\rho - b(\rho)\frac{\partial}{\partial t} + \sum_{i=1}^{n} p_i(y,t)\frac{\partial}{\partial y_i} + p_0,$$

where M_ρ is an operator containing only differentiation on the unit sphere. In this case, for functions w with the properties: 1) $w(y,t)$ have continuous derivatives of second order with respect to y_i and of first order with respect to t; 2) $w \equiv 0$ in $\Pi_\varepsilon^{*[0,T]}$ for some positive ε; 3) $w|_{\rho=s} = \partial w/\partial \rho|_{\rho=s} = 0$; 4) $w|_{t=0} = w|_{t=T} = 0$, the inequality can be obtained

$$\int_{\Pi_s^{*[0,T]}} \frac{w^2}{\rho^{2\beta+n-2}}\, dy\, dt \leq \frac{2}{\beta^2(\beta+n-3)} \int_{\Pi_s^{*[0,T]}} \frac{(H^* w)^2}{\rho^{2\beta+n-4}}\, dy\, dt. \tag{4}$$

Let $u^*(y,t) \equiv u[x(y),t]$ be a solution of the equation $H^* u^* = 0$, satisfying the equalities (2), where $M = 1$. We multiply $u^*(y,t)$ by the cutoff functions $f(\rho)$ and $\varphi(t)$ so that the function $w \equiv u^*(y,t) f(\rho)\varphi(t)$ has properties 1)–4), and $w \equiv u^*(y,t)$ in the domain $G: \rho_0 \leq |y| \leq s'/2,\ a/2 \leq t \leq T - a/2$. Taking into account that, for sufficiently large β, $(\rho_0/2)^\beta \leq \Delta$, from inequality (4) one can obtain the inequality

$$\int_G \left(\frac{u^{*2}}{\rho^{2\beta+n-2}}\right) dy\, dt \leq \left(\frac{2}{s}\right)^{2\beta-1}. \tag{5}$$

Hence we have

$$\frac{1}{\tilde{\rho}^{\,n-1}} \int_{\Pi^*_{\tilde\rho}[a/2, T-a/2]} u^{*2}\, dy\, dt \leq (C_1 \tilde{\rho})^{2\beta-1}$$

for any $\tilde{\rho}$, $\rho_0 < \tilde{\rho} < 1/2C_1$. Returning, by means of the inverse transformation, to the variables $(x_1, \dots, x_n, t)$, we obtain

$$\frac{1}{\tilde{r}^{\,n-1}} \int_{\Pi_{\tilde r}^{[a/2, T-a/2]}} u^2\, dx\, dt \leq (C_2 \tilde{r})^{2\beta-1} \tag{6}$$

for all $\tilde{r}$, $r_0 < \tilde{r} \leq R/2C_2$. Let now $\tilde{r}$ be such that $2r_0 < \tilde{r} < R/2C_2$. By the mean-value theorem for the integrals

$$\int_{a/2}^{3a/4} \int_{D_t} u^2 \, dx \, dt,$$

$$\int_{3\tilde{r}/4}^{\tilde{r}} \int_{K_1} \int_0^T u^2 r^{n-1} \, dr \, dO \, dt$$

(K_1 is the unit sphere and dO is the element of the surface of K_1), there exist r_1 and t_1 such that $3\tilde{r}/4 \leq r_1 \leq \tilde{r}$, $a/2 \leq t_1 \leq 3a/4$, and

$$\int_{K_1} \int_0^T u^2\big|_{r=r_1} r_1^{n-1} \, dO \, dt \leq \frac{4}{\tilde{r}} (C_2\tilde{r})^{2\beta-1} (\tilde{r}^{\,n-1}); \tag{7}$$

$$\int_0^r \int_{K_1} u^2\big|_{t=t_1} r^{n-1} \, dr \, dO \leq \frac{4}{a} (C_2\tilde{r})^{2\beta-1} (\tilde{r}^{\,n-1}). \tag{8}$$

For any point $(x, t) \in \Pi_{\tilde{r}/2}^{[a,T-a]}$, the solution $u(x, t)$ is representable in the form

$$u(x,t) = \int_{\Pi_1} \Gamma_1(x,t,\xi,\tau) u(\xi,\tau) r_1^{n-1} \, dO \, d\tau + \int_{\Pi_2} \Gamma_2(x,t,\xi,t_1) u(\xi,t_1) \, d\xi, \tag{9}$$

where

$$\Pi_1 = \{(x,t) \mid |x| = r_1, \; t_1 \leq t \leq T - a\};$$

$$\Pi_2 = \{(x,t) \mid 0 \leq |x| \leq r_1, \; t = t_1\}.$$

$\Gamma_1(x, t, \xi, \tau)$ and $\Gamma_2(x, t, \xi, \tau)$ are Green' s functions. Using Schwarz' s inequality, (7) and (8), as well as the known estimates for Green' s functions (see (3, 4)), we obtain $|u(x, t)| \leq (Cr)^{\beta-1}$ for all $\tilde{r}$, $r_0 \leq \tilde{r} \leq R/2C$, and t, $a \leq t \leq T - a$; C is a constant depending on a, n, and the equation; $\beta = \ln \Delta / \ln(Br_0)$; $2B = \max_{\rho_0/2 \leq \tilde{\rho} \leq 1/2C_1} (\tilde{\rho}/\tilde{r})$.

Let $M(R, 0, T) = M \neq 1$. Introduce the function $v = u/M$. By what has been proved earlier, $M_v(r, a, T-a) \leq (Cr)^{\beta-1}$, where $\beta = \ln(\Delta/M)/\ln(Br_0)$. Taking into account that $M_u(r, a, T - a) = M \cdot M_v(r, a, T - a)$ and taking logarithms, we obtain the assertion of the theorem.

Corollary. Let $r_0 = R/4C$, $r = R/2C$, where C is the constant of Theorem 1, $M_u(r_0, 0, T) = r_0^{\gamma}$; $M_u(R, 0, T) = M$. Then the inequality $M(r, a, T - a) < r^{\gamma'}$ is valid, where $\gamma' = \gamma\alpha_1$, $\alpha_1 = \ln(R/2)/2\ln(BR/4C)$.

Lemma 1. Let $u(x, t)$ be a solution of equation (1), satisfying the conditions

$$M_u(r_0, 0, T) = \Delta, \qquad M_u(R, 0, T) = M. \tag{10}$$

For any $\varepsilon > 0$ the inequality

$$M_u(R - \varepsilon, t_1, t_2) < (R/4C)^{\gamma\omega^N}, \tag{11}$$

holds, where

$$N = \left[\ln \frac{2C\varepsilon}{R(2C-1)} \Big/ \ln \frac{2(2C-1)}{4C-1}\right]; \qquad t_1 = a\sum_{p=0}^{N}\left[\frac{2(2C-1)}{4C-1}\right]^{2p};$$

$$t_2 = T - t_1;$$

$$\gamma = \frac{\ln\Delta \cdot \ln(R/4)}{2\ln(Br_0)\ln(R/4C)}, \qquad \omega = \frac{\ln(R/4C)}{\ln(R/2C)} \cdot \frac{\ln(R/2)}{2\ln(Br/4C)};$$

B, C are the constants of Theorem 1.

Proof. By Theorem 1, for $\tilde{r} = R/4C$ we have

$$M_u(R/4C, a, T{-}a) \le (R/4)^{(\ln\Delta/\ln(Br_0))-1+\ln M\cdot\{1/\ln(R/4)-1/\ln(Br_0)\}\gamma} < (R/4C)^{\tilde{\gamma}},$$

where

$$\gamma = \frac{\ln\Delta}{2\ln(Br_0)}\frac{\ln(R/4)}{\ln(R/4C)}.$$

By the corollary to Theorem 1, for $r = R/2C$ we have

$$M_u(R/2C, 2a, T - 2a) \le (R/2C)^{\gamma\alpha_1}.$$

Denote by $K_{\bar{O}}^{\bar{r}}$ the ball in the space $\{x\}$ with center at $\bar{O}$ and radius $\bar{r}$, and set

$$M_u^{\bar{O}}(\bar{r}, \bar{t}_1, \bar{t}_2) = \max_{(x,t)\in K_{\bar{O}}^{\bar{r}}\times[\bar{t}_1,\bar{t}_2]} |u(x,t)|.$$

In $K_O^{R/2C}$ inscribe a ball $K_{O_1}^{r_1}$ tangent to it from inside, and in K_O^R a ball $K_{O_1}^{R_1}$ tangent to it from inside, so that $r_1 = R_1/4C$.

Make the change of coordinates

$$x'_j = k_1[x_j - (x_j)_{O_1}], \qquad t' = k_1^2 t,$$

where

$$k_1 = R/R_1 = (4C-1)/2(2C-1).$$

Under this transformation the equation $Hu = 0$ becomes $H_1 u_1$, where

$$H_1 \equiv \sum_{i,k=1}^{n} a_{ik}\frac{\partial^2}{\partial x'_i \partial x'_k} + \sum_{i=1}^{n}\frac{b_i}{k_1}\frac{\partial}{\partial x'_i} + \frac{c}{k_1^2} - \frac{\partial}{\partial t'},$$

with coefficients satisfying the same conditions as the coefficients of the operator H, while

$$u_1(x',t') \equiv u[x'(x),t'(t)]$$

will satisfy the conditions

$$M_{u_1}^{O_1}(R/4C;k_1^2{\cdot}2a;k_1^2(T{-}2a)) = M_u^{O_1}(R_1/4C;2a;T{-}2a) \leq M_u^{O}(R/2C;2a;T{-}2a) < (R/4C)^{\gamma\alpha_1\alpha_2},$$

$$M_{u_1}^{O_1}(R;k_1^2 2a;k_1^2(T-2a)) = M_u^{O_1}(R_1;2a;T-2a) \leq M_u^{O}(R;2a;T-2a) \leq M,$$

where

$$\alpha_2 = \frac{\ln(R/4C)}{\ln(R/2C)}.$$

Then the corollary to Theorem 1 is applicable to u_1. We have

$$M_{u_1}^{O_1}(R/2C;k_1^2{\cdot}2a{+}2a;k_1^2(T{-}2a){-}2a) = M_u^{O_1}(R/2C;t_1^{(1)};t_2^{(1)}) \leq (R/2C)^{\gamma\alpha_1^2\alpha_2} = (R/4C)^{\gamma\omega^2},$$

where

$$\omega = \alpha_1\alpha_2, \qquad t_1^{(1)} = 2a(1+1/k_1^2), \qquad t_2^{(1)} = T - t_1^{(1)}.$$

Carrying out an analogous construction for the balls $K_{O_1}^{R/2C}$ and $K_{O_1}^{R_1}$, by the same method we obtain the estimate

$$M_u^{O_2}(R_2/2C;t_1^{(2)};t_2^{(2)}) \leq (R/4C)^{\gamma\omega^3}.$$

Let N be the number of steps of this process for which

$$(1-1/2C)R_{N-1} > \varepsilon \geq (1-1/2C)R_N,$$

i.e.

$$N = \left[\ln\frac{2C\varepsilon}{R(2C-1)}\Big/\ln\frac{2(2C-1)}{4C-1}\right].$$

Then the inequality

$$M_u^O(R-\varepsilon,t_1,t_2) < (R/4C)^{\gamma\omega^N}$$

is valid, where

$$t_1 = 2a\sum_{p=0}^{N}\left[\frac{2(2C-1)}{4C-1}\right]^{2p}, \qquad t_2 = T - t_1,$$

which was required to be proved.

Lemma 2. Let $u(x,t)$ be a solution of equation (1) in $\Pi_R^{[0,T]}$, satisfying the conditions: $M_u(R,0,T) = M$; $M_u(R-\varepsilon,t_1,T-t_1) < \delta$; $u|_{|x|=R} < \delta$. Let

$$m(t^*) = \max_{R-\varepsilon\leq|x|\leq R,\ t=t^*}|u(x,t)|.$$

Then for any $t \in (t_1, T - t_1)$

$$m(t) \le 2 \max \left\{ \delta \exp Kt;\ M\sqrt{2} \exp \left[(K - \mu/\varepsilon^2)(t - t_1) \right] \right\},$$

where K, μ are constants from conditions 2) and 3) imposed on the coefficients of equation (1).

The proof is carried out by means of constructing a barrier.

Theorem 2 (on two cylinders). Let $u(x,t)$ be a solution of equation (1) in $\Pi_R^{[0,T]}$, satisfying the conditions: $M_u(R, 0, T) = M$; $M_u(r_0, 0, T) = \Delta$; $u|_{|x|=R} = 0$. Then for any r, $r_0 < r < R$,

$$\ln M_u(r; T/4; T) \le C_1 \left[\ln \Delta / \ln(Br_0) \right]^{\theta},$$

where

$$1/\theta = 1 + 2C \ln \left[\ln(4C/B) / \ln \sqrt{2} \right];$$

C, B are the constants of Theorem 1; C_1 is a constant depending on a, n, and the equation.

Proof. By Lemma 1, for any $\varepsilon > 0$ inequality (11) is valid. Choosing a so small that $t_1 < T/8$, and setting $\delta = (R/4C)^{\gamma\omega^N}$, by Lemma 2 we obtain

$$M_u(r; T/4; T) \le \min_{\varepsilon > 0} \left(2 \max \left\{ (R/4C)^{\gamma\omega^N} \exp(3KT/4);\ M\sqrt{2} \exp \left[(K - \mu/\varepsilon^2)(T/4) \right] \right\} \right) \tag{12}$$

for any $r \le R$.

From monotonicity considerations for the quantities in the braces in inequality (12), it is clear that the optimal value $\varepsilon = \varepsilon_0$ is obtained in the case of equality of these quantities. Taking a value close to ε_0,

$$\varepsilon = \frac{(\ln(Br_0))\mu TR}{(2C - 1)\, 16C \, \ln \Delta \cdot \ln(R/4C)},$$

we obtain the estimate

$$M_u(r; T/4; T) < (\theta_1)^{[\ln \Delta / \ln(Br_0)]^{\theta}},$$

where $\theta_1 < 1$ depends on a, n, and the equation,

$$1/\theta = 1 + 2C \ln \left[\ln(4C/B) / \ln \sqrt{2} \right];$$

C and B are the constants of Theorem 1. The assertion of Theorem 2 follows from this.

We give two applications of Theorems 1 and 2 proved above.

Theorem 3 (an analogue of Ito's problem). Let $u(x,t)$ be a solution of equation (1) in $\Pi_R^{[-T,0]}$, satisfying the conditions

$$M_u(r_0, -\tau, 0) = \Delta; \qquad M_u(R, -T, 0) = M; \quad u|_{|x|=R} = 0.$$

Then

$$M(R,-T+1,0) \le A_1 \theta_1^{A_2[\ln\Delta/\ln(Br_0)]^\theta},$$

where θ is the constant of Theorem 2. $\theta_1 < 1$ is a constant depending on τ, n, and the equation.

The proof is obtained from Theorem 2 of the present article and Theorem 2 of paper (5).

Theorem 4. (Possible rate of decrease of a solution of equation (1) in a neighborhood of an irregular boundary point.) Let $\Pi = D \times [0,T]$, where D is some domain in the space of x. Let x^0 be a boundary point of the domain D such that there exists a circular cone with vertex at x^0, lying entirely in the domain D. Let φ be the angle made by the generatrix with the axis. Let $u(x,t)$ be a solution of equation (1) in Π. Put

$$M(r) = \max_{|x-x_0|\le r,\ (x,t)\in\Pi} |u(x,t)|.$$

If

$$\overline{\lim_{r\to 0}} \left[M(r)\exp\left(C_1/r^{C_2/\varphi}\right)\right],$$

where C_1, C_2 are constants depending on the equation, then $u \equiv 0$ in Π.

This assertion is a consequence of Theorem 1 (on three cylinders). Its proof is similar to the proof of Lemma 1.

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REFERENCES

1. A. I. Markushevich, *Theory of Analytic Functions*, Moscow-Leningrad, 1950.

2. L. I. Danilyan, DAN, **129**, No. 5, 979 (1959).

3. W. Pogorzelski, Ann. Polon. Math., **4**, No. 1, 61 (1957); **4**, No. 1, 110 (1957).

4. W. Pogorzelski, Ann. Polon. Math., **4**, No. 3, 288 (1958).

5. R. Ya. Glagoleva, DAN, **148**, No. 1, 20 (1963).