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Abstract

Full Text

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APPROXIMATIVELY CONVEX SETS IN BANACH SPACES

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In the present article the concept of a Chebyshev set, introduced in ([1]), is generalized, and several theorems are proved on the connection of generalized Chebyshev (otherwise: approximately convex) sets with convex sets. These theorems are an extension of the author' s results concerning Chebyshev sets ([6]) to approximately convex sets.

Let X be a linear normed space, and let T be a multivalued mapping of X into itself, more precisely, a single-valued mapping of X into the set of subsets of X. We shall call such a mapping convex-valued if, for every $x \in X$, the set Tx is nonempty and convex. T is called continuous here if from $x_n \to x$, $y_n \to y$, $y_n \in Tx_n$, it follows that $y \in Tx$. Let M be a set in X. In what follows we shall denote by T the operator of metric projection (metric projection) of the space X onto the set M, i.e., such a mapping which assigns to each point $x \in X$ the set Tx of all points $y \in M$ satisfying the condition $\rho(x, y) = \rho(x, M)$ (abbreviated: $xy = xM$). If $y \in Tx$, then one says that x is projected into y. T will be continuous if and only if M is closed. In the case when T is a convex-valued operator, we call the set M **approximately convex**. It is clear that an approximately convex set is always closed, and the operator T for such a set is continuous. If, in particular, T is single-valued, then M is called a Chebyshev set ([1]). Recall that the segment $[x, y]$ (respectively, the ray $\overrightarrow{xy}$) is the set of points of the form $(1-\lambda)x + \lambda y$, where λ runs through the interval $[0, 1]$ (respectively, the ray $[0, \infty)$). Generalizing the notion introduced in ([1]), we shall call a set M a **sun** if for every $x \in X$ there exists a point $y \in Tx$ such that all points of the ray $\overrightarrow{yx}$ are projected into y. Let us note that the points of the segment $[y, x]$ are always projected into y, if $y \in Tx$, as the following proposition shows:

1°. If $xy = xM$, $y \in M$, $z \in [x, y]$, then $zy = zM$.

We shall say that a point x' **majorizes** a point x, and write $x' \succ x$, if there exists $y \in Tx'$ such that $x \in [x', y]$. For example, if $y \in Tx$, then $x \succ y$.

2°. If $x'' \succ x$, $x' \in [x'', x]$, then $x'' \succ x'$, $x' \succ x$.

3°. If $x' \succ x$, $x' \neq x$, then $x'M > xM$.

4°. If M is approximately convex, then the relation $\succ$ is transitive.

Proof. Let $x'' \succ x'$, $x' \succ x$; then
$\alpha)$ $x'y = x'M$;
$\beta)$ $x \in [y, x']$;
$\gamma)$ $x''y' = x''M$;
$\delta)$ $x' \in [y', x'']$.

From α, β, and 1° it follows that
$\varepsilon)$ $xy = xM$.

From γ, δ, and 1° it follows that
$\zeta)$ $x'y' = x'M$.

Extend the line $x''x$ to its intersection with the line yy' at the point y''. (It is easy to see that the points x, x', x'', y, y' lie in one plane and that $y'' \in [y, y']$.) We have:

$$x''M \leq x''y \leq x''x' + x'y \overset{(\alpha)}{=} x''x' + x'M \overset{(\zeta)}{=} x''x' + x'y' \overset{(8)}{=} x''y' \overset{(\gamma)}{=} x''M.$$

Hence $x''y = x''M = x''y'$. This gives $y \in Tx''$, $y' \in Tx''$, and, by the approximative convexity of M, $[y, y'] \subset Tx''$ and $y'' \in Tx''$. Taking into account that, by construction, $x \in [x'', y'']$, we infer $x'' \succ x$, and 4^0 is proved.

Let $x \in X$. Denote by K_x the set of all points majorizing x. Recall that a set is called boundedly compact (^{2}) if its intersection with every closed ball is compact.

5^0. *For every $x \in X \setminus M$, the set K_x is boundedly compact, provided the set Tx is compact.*

Proof. K_x is contained in the set D of all points of the form $(1-\lambda)y + \lambda x$, where λ runs through the ray $[1, \infty)$, and y through the set Tx. D is obviously boundedly compact; therefore it suffices to show that K_x is closed. Let $x'_n \in K_x$, $x'_n \to x'$. There exists $y_n \in Tx'_n$ such that

$$x = (1-\lambda_n)x'_n + \lambda_n y_n, \quad 0 \leq \lambda_n \leq 1.$$

Choosing subsequences $\lambda_{n_k} \to \lambda$, $y_{n_k} \to y \in M$, we obtain: $x_{n_k} \to x'$, $y_{n_k} \to y$, $y_{n_k} \in Tx'_{n_k}$. Hence, by the continuity of T, we have $y \in Tx'$. Passing to the limit in the equality $x = (1-\lambda_n)x'_n + \lambda_n y_n$, we obtain $x \in [x', y]$, and 5^0 is established.

For what follows we shall need the following assertion:

6^0. *In a Banach space, every convex-valued continuous mapping of a convex compact set into itself has a fixed point.*

By a fixed point of a multivalued mapping R is meant a point x_0 such that $x_0 \in Rx_0$.

The validity of this theorem follows from the following, more general proposition (taking into account 8^0 and 9^0, and the obvious fact that every compact convex set is contractible to a point; for the substance of the terms occurring in the references, see ([3],[5])).

7^0. **Theorem** ([3]). *Let M be a compact metric space which is an acyclic absolute neighborhood retract, and let $T : M \to M$ be a continuous multivalued function such that, for every $x \in M$, the set Tx is acyclic. Then T has a fixed point.*

8^0. **Theorem** ([4]). *A separable and convex subset of a linear normed space is an absolute retract (and hence an absolute neighborhood retract).*

9^0. **Theorem** ([3]). *Every space contractible to a point is homologically trivial (i.e., acyclic).*

10^0. *In a Banach space, every convex-valued continuous mapping R of a convex closed set V into its compact part has a fixed point.*

For the proof it suffices to apply 6^0 to the compact set $\widetilde{V}$, the closed convex hull of the set RV, and to the mapping $\widetilde{R}$, the restriction of R to $\widetilde{V}$.

11^0 **Theorem.** *Every approximatively convex and boundedly compact set M of a Banach space X is a sun.*

First we establish the following property of the set M:

12^0. *For every $x \in X \setminus M$, the set $K_x \setminus \{x\}$ is nonempty.*

Since M is closed, there exists a closed ball V with center at the point x and radius $r > 0$ such that the following relation holds: for every $z \in V$,

$$zM \geq \rho(V, M) = \inf_{x \in V,\, y \in M} \rho(x, y) > r.$$

Put, for $z \in V$,

$$Rz = \frac{r}{zM}(z - Tz) + x.$$

Since the functional zM is continuous and nonzero for $z \in V$, and T is continuous, it is not hard to see that R will also be continuous. R maps V into itself, since $\|Rz - x\| = r$. The set RV is relatively compact, because the set $TV \subset M$ is such. Finally, since Rz is linearly related to Tz, the convex-

sets Tz pass into convex sets Rz. Thus, R satisfies all the conditions of Theorem 10°. Consequently, there is a point $x_0 \in V$ for which $x_0 \in Rx_0$, i.e., there exists $y \in Tx_0$ such that

$$x_0 = x + \frac{r}{x_0 M}(x_0 - y).$$

Hence

$$x_0 = \left(-\frac{r}{x_0 M} y + x\right) \frac{1}{1 - r/x_0 M} = (1-\lambda)y + \lambda x = x_\lambda,$$

where $\lambda = \dfrac{1}{1 - r/x_0 M} > 1$, i.e. $x_\lambda > x,\ x_\lambda \neq x$. This proves property 12°.

Let us pass to the proof of 11°. Let x be an arbitrary point of the set $X \setminus M$. If K_x were bounded, then it would be compact (by 5°; here the compactness of Tx follows from the bounded compactness of M); therefore K_x would contain a point x', farthest from M. By 12° there exists $x'' > x'$, $x'' \neq x'$, such that, by 3°, $x''M > x'M$ and, by 4°, $x'' \in K_x$, which contradicts the choice of the point x'. Consequently, K_x is unbounded. The theorem will evidently be proved if we show that K_x contains a ray emanating from the point x. Let $x'_n \in K_x$ and $\|x'_n\| \to \infty$. Put

$$z_n = x + \frac{x'_n - x}{\|x'_n - x\|}.$$

Then $\|z_n - x\| = 1,\ z_n \in [x, x'_n]$, so that, by 2°, $z_n \in K_x$, and, by 5°, it has a limit point $z \in K_x$. We shall assume that already $z_n \to z$. We shall show that the ray $\overrightarrow{xz}$ is contained in K_x. Let $u \in \overrightarrow{xz}$, i.e. $u = x + A(z - x)$, where $A \geq 0$. Put

$$u_n = x + A\frac{x'_n - x}{\|x'_n - x\|} = x + A(z_n - x).$$

Then $u_n \to u$, and

$$u_n = \left(1 - \frac{A}{\|x'_n - x\|}\right) x + \frac{A}{\|x'_n - x\|} x'_n,$$

therefore, for sufficiently large n, $u_n \in [x, x'_n]$ and $u_n \in K_x$. Hence $u \in K_x$, as was required.

13°. **Theorem.** *In a smooth linear normed space every sun is a convex set.*

Proof. Suppose the contrary, that the sun M is not convex. Then there exist points $a \in M,\ b \in M,\ x \in [a, b] \setminus M$. Take $y \in Tx$. Suppose that the segments $[a, y]$ and $[b, y]$ do not intersect the interior of the ball V with center at x and radius $\|x - y\|$. Then the corresponding straight lines ay and by also do not intersect the interior of V. If, for example, we had $\|d\| < \|x - y\|,\ y \in [d, b]$, then the point of intersection of the segments $[a, y]$ and $[x, d]$ (obviously, they lie in one plane) would be interior to V. By the Hahn-Banach theorem, there exist at the point y supporting hyperplanes to the ball V, passing respectively through ay and by, and therefore distinct. This contradicts the assumed smoothness of the space. Thus, for every $y \in Tx$ one of the segments $[a, y]$, $[b, y]$ intersects the interior of the ball V. By the definition of a sun, there is such a $y \in Tx$ that $(\overrightarrow{yx} \setminus [y, x]) \subset K_x$. Suppose, for example, that the segment $[a, y]$ intersects the interior of the ball V. Then for some ball V', similar to the ball V with center of similarity at the point y, the point a is interior. It is clear that the center x' of the ball V' majorizes x, but, on the other hand, the point $a \in M$ is closer to x' than y. This is impossible. The theorem is proved.

From 11° and 13° it follows that

14°. **Theorem.** *In a smooth Banach space every approximately convex and boundedly compact set is convex.*

15°. **Theorem.** *In an n-dimensional Banach space every approximately convex set is convex if and only if the space is smooth.*

Sufficiency is given by Theorem 14°.

Necessity. If y is a point of nonsmoothness of the unit sphere, then as the desired approximately convex but nonconvex set one may take the union of two distinct closed supporting half-spaces at the point y, not containing the unit ball and intersecting with it in a common set (obviously convex).

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Note: Figure translations are in progress. See original paper for figures.