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Abstract

Full Text

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Hartogs' Theorem in Certain Normed Fields

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Introduction. One of the fundamental theorems in the theory of analytic functions of several complex variables is Hartogs' fundamental theorem (see ([1])). The question arises whether this theorem holds for power series over other fields. In the present note a positive answer to this question is given for certain normed fields.

Let H be a field complete with respect to a non-Archimedean norm of rank 1 defined in it. We shall regard the group G of values of the norm in multiplicative notation. G is a subgroup of the multiplicative group of positive real numbers. The norm of an element $x \in H$ will be denoted by $|x|$. As usual, we adjoin zero to G, putting $|0| = 0$. Recall the following properties of the norm: for any $x, y \in H$, $|xy| = |x| \cdot |y|$, $|x+y| \leq \max(|x|, |y|)$, and in the case $|x| \neq |y|$ equality holds in the latter relation.

Suppose that the field H is not locally compact. Then it has at least one of the following two properties:

Property 1. The group G of values of the norm is dense with respect to the ordinary absolute value.

Property 2. The residue field of the normed field H contains an infinite number of elements.

In such a field Hartogs' theorem holds. To simplify the proof, suppose that H is algebraically closed. Then properties 1 and 2 are satisfied simultaneously. We introduce several further definitions and notations.

We shall call a **disk** with center at the point $x_0 \in H$ and radius r the set $\{x \in H,\ |x - x_0| < r\}$. The **closed disk** and **circle** are defined correspondingly. We shall call a **polydisk** the topological product of several disks. A function $f(x)$ over the field H will be called **holomorphic** in a given disk if it is the sum of a power series convergent in this disk. A function of several variables will be called **holomorphic** if it can be represented as the sum of a multiple power series with coefficients in H.

We denote by $M_f(r)$ the maximum of the norm of the holomorphic function $f(x)$ on the circle $|x| = r$. This maximum, by virtue of property 2 of the field H, is always equal to the norm of the maximal term of the power series $f(x)$ (see ([2])). The usual maximum principle holds (see ([2], [3])). The Cauchy inequality for the coefficients of a power series also holds.*

Lemma 1. *Let $f(x)$ be holomorphic in the disk $|x| < R$, $f(0) \neq 0$. Denote by $n(t)$ the number of roots of $f(x)$ in the closed disk $|x| \leq t$. Then for any $r < R$*

$$\int_0^r \frac{n(t)}{t}\,dt = \ln M_f(r) - \ln|f(0)|.$$

* All the basic properties of power series over H are easily extended to power series over the field of complex numbers, in particular the concept of conjugate radii of convergence, the property of logarithmic convexity of the domain of convergence, and also the existence of a power series with a given logarithmically convex domain of convergence.

This lemma coincides with Theorem 2 in ([4]).

Lemma 2. Denote by Γ_R the circumference $|x| = R$. Whatever the points $a_1, \dots, a_n \in \Gamma_R$ may be (among which there may also be coincident ones), and whatever $r \leq R$ may be, one can find on Γ_r such a finite system of disks with sum of radii equal to r that outside these disks the inequality

$$\prod_{j=1}^{n} |x - a_j| > \left(\frac{r}{e}\right)^n$$

will hold.

Moreover, the minimal radius of these disks is not less than r/n.

This lemma is analogous to Cartan' s theorem (see ([5]), Ch. I, § 7); the proof is the same. We note only that the radii of the disks under consideration do not exceed r, and therefore they all belong to Γ_R.

Theorem. Let the function $f(x_1, \dots, x_n)$ be holomorphic in each of the variables (with the others fixed) in the polydisk $|x_i| < R_i$ $(i = 1, \dots, n)$. Then $f(x_1, \dots, x_n)$ is holomorphic in this polydisk.

We divide the proof into several parts.

1°. For brevity we introduce the notation: $k = (k_1, \dots, k_n)$; $y = y_1, \dots, y_n$; $y^k = y_1^{k_1} \dots y_n^{k_n}$; $\|k\| = k_1 + \dots + k_n$; $|y| \leq \rho$ means $|y_i| \leq \rho$ $(i = 1, \dots, n)$.

Applying induction, suppose that the theorem is true for n variables, and consider the function $f(x, y) = f(x, y_1, \dots, y_n)$. It is known that it is holomorphic in each of the variables in the polydisk

$$|x| < R, \qquad |y_i| < R_i \quad (i = 1, \dots, n). \tag{1}$$

From the induction hypothesis it follows that $f(x, y)$ expands in a series of the form

$$f(x,y)=\sum_k f_k(x)y^k=\sum_{k_1,\ldots,k_n} f_{k_1,\ldots,k_n}(x)y_1^{k_1}\ldots y_n^{k_n}, \tag{2}$$

convergent in the polydisk (1).

Take arbitrary values of the norm $R'<R$, $R_i'<R_i$ $(i=1,\ldots,n)$. Without loss of generality, one may assume $R_i'=1$. Thus $f(x,y)$ satisfies the conditions of the theorem in the closed polydisk

$$|x|\le R',\qquad |y|\le 1. \tag{3}$$

Hence, as in Osgood' s lemma, we obtain boundedness of $f(x,y)$ in the interior polydisk. The properties of the non-Archimedean norm make it possible to move the centers of the disks constituting it to the origin of coordinates. Obviously, one may assume that in the indicated polydisk $|f(x,y)|\le 1$. Then from (2), with the aid of Cauchy' s inequality, we obtain

$$|f_k(x)|\le A^{\|k\|},\qquad |x|\le R';\qquad A \text{ is a constant.} \tag{4}$$

From this inequality, using the induction hypothesis, we obtain the holomorphy of $f_k(x)$ for $|x|<R$. Finally, one may assume that for all k

$$|f_k(0)|\ge e^{-\|k\|}. \tag{5}$$

If this is not fulfilled for the sets of indices $k_1,k_2,\ldots$, then it suffices to add to $f(x,y)$ the series $\sum_i a^{-\|k_i\|}y^{k_i}$, $a\in H$, $2\le|a|\le e$.

2°. Denote by $n_k(t)$ the number of zeros of $f_k(x)$ in the disk $|x|\le t$. From Lemma 1 and inequalities (4) and (5) one can obtain, for $R''<R'$, the inequality

$$n_k(R'')\le\frac{R'}{R'-R''}(\|k\|\ln A+\|k\|)=B\|k\|;\qquad B \text{ is a constant.} \tag{6}$$

3°. Let α be an arbitrary positive number. We shall show that for all sufficiently large $\|k\|$ one has

$$|f_k(x)|\leqslant e^{\alpha\|k\|},\qquad |x|\leqslant R'''<R', \tag{7}$$

where R''' is arbitrary but fixed. If (7) is proved, then the completion of the proof presents no difficulty. Indeed, write $f_k(x)$ in the form

$$f_k(x)=\sum_{i=0}^{\infty}a_{k,i}x^i=\sum_{i=0}^{\infty}a_{k_1,\ldots,k_n,i}x^i.$$

Taking into account that α in (7) is arbitrarily small, it is easy to obtain the convergence of the series

$$\sum_{k,i} a_{k,i} x^i y^k = \sum_k f_k(x) y^k = f(x,y)$$

in any polydisc interior with respect to (3), and consequently in the whole polydisc (1). Thus, everything is reduced to the proof of (7).

4°. Suppose that (7) is not satisfied. Then there exists an infinite sequence

$$k_1, k_2, \dots, k_i, \dots; \qquad \|k_i\| < \|k_{i+1}\|, \tag{8}$$

for which

$$M_{f_{k_i}}(\rho) > e^{\alpha \|k_i\|}, \qquad R''' \leqslant \rho \leqslant R''. \tag{9}$$

Choose an arbitrary $\varepsilon > 0$. Select in the annulus $R''' \leqslant |x| \leqslant R''$ N normed circles (i.e. such circles whose radii are values of the norm), where $N > B/\varepsilon$, B being defined in (6). This is possible by property 1 of the field H. It follows from (6) that for each $f_{k_i}(x)$, on at least one of these circles the number of roots does not exceed the number $\varepsilon\|k_i\|$. Choose from our circles one for which the latter occurs infinitely many times. We may assume that this occurs for the whole sequence (8). We may also assume, without loss of generality, that the chosen circle has radius 1. Denote it by Γ_1. Thus, on Γ_1, for all $f_{k_i}(x)$, the number of roots s_i satisfies the inequality

$$s_i < \varepsilon \|k_i\|. \tag{10}$$

Let $k = k_i$ be one of such sets of indices. Write $f_k(x)$ in the form

$$f_k(x) = g_k(x) \prod_{j=1}^{s} (x - a_j), \tag{11}$$

where $a_1, \dots, a_s$ are all the roots of $f_k(x)$ on Γ_1; $g_k(x)$ does not vanish anywhere on Γ_1, and therefore has a unique maximal term on Γ_1 and preserves a constant norm there. Taking this into account, from (9) we easily obtain

$$|g_k(x)| > e^{\alpha \|k\|}, \qquad x \in \Gamma_1. \tag{12}$$

Denote by E_k the set of those $x \in \Gamma_1$ for which $|f_k(x)| \leqslant 1$. Then from (11) and (12) we obtain

$$\prod_{j=1}^{s} |x - a_j| < e^{-\alpha\|k\|}, \qquad x \in E_k. \tag{13}$$

According to Lemma 2, on Γ_1 there exists a finite set of discs U_k

with sum of radii equal to $r < 1$, outside which

$$\prod_{j=1}^{s} |x - a_j| > (r/e)^s.$$

Using (10), we obtain

$$\prod_{j=1}^{s} |x - a_j| > \left(\frac{r}{e}\right)^{\varepsilon\|k\|}.$$

We choose ε so small that $(r/e)^\varepsilon > e^{-a}$. Comparing the latter with (13), we see that E_k is entirely covered by the set U_k. Put

$$D_m = \bigcup_{i=m}^{\infty} E_{k_i}.$$

Then D_m has a finite covering U_{k_m}. Moreover, $D_m \subset D_{m+1}$. We also note that every point of Γ_1, by virtue of the convergence of the series (2) for $y = 1$, must belong to some D_m; consequently,

$$\bigcup_{m=1}^{\infty} D_m = \Gamma_1.$$

On the other hand, it follows from the properties of the sets D_m that their sum does not exhaust the whole circumference Γ_1 (the proof of this fact is somewhat complicated by the fact that the field H is not locally compact). From this contradiction follows the validity of inequality (7), which completes the proof of the theorem.

Remark. If the residue field of the field H contains an uncountable set of elements, then Hartogs' theorem is proved much more simply. After 1° we argue as follows: from the circumference $\Gamma_{R'}$ of radius R' with center at zero we remove neighborhoods (open) of radius R' with centers at all roots $f_k(x)$ from (2). There are countably many such neighborhoods; consequently, they will not cover the whole $\Gamma_{R'}$. At the remaining points $x \in \Gamma_{R'}$ we have $|f_k(x)| = M_{f_k}(R')$ for all k; hence the convergence of the series at such a point means uniform convergence of the series on $\Gamma_{R'}$, and from this it is not difficult to conclude that $f(x, y)$ is holomorphic.

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Note: Figure translations are in progress. See original paper for figures.