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Abstract

Full Text

Mathematical Physics

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Solution of the Dirichlet Problem for a Finite Wedge by Means of a Special Integral Transform with Respect to Cylindrical Functions

(Presented by Academician B. P. Konstantinov, February 4, 1965)

1°. In the present note, an integral transform with respect to cylindrical functions of imaginary order and argument is used to obtain an exact solution of the Dirichlet problem for the domain bounded by the cylindrical surface $r = a$ and the planes $\theta = \theta_1$, $\theta = \theta_2$, $z = 0$, $z = l$ (r, θ, z are cylindrical coordinates).

We seek a function $u(r, \theta, z)$, harmonic in the indicated domain, bounded as $r \rightarrow 0$, and satisfying the boundary conditions

$$u|_{z=0} = u|_{z=l} = u|_{r=a} = 0, \quad u|_{\theta=\theta_1} = \Phi_1(r, z), \quad u|_{\theta=\theta_2} = \Phi_2(r, z), \quad (1)$$

in the form of the series

$$u(r, \theta, z) = \sum_{n=1}^{\infty} u_n(r, \theta) \sin n\pi \frac{z}{l}. \quad (2)$$

In this case the functions u_n must satisfy the equation

$$\frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial u_n}{\partial x} \right) + \frac{1}{x^2} \frac{\partial^2 u_n}{\partial \theta^2} - u_n = 0, \quad x = \frac{n\pi r}{l}, \quad (3)$$

and the conditions

$$u_n(a, \theta) = 0, \quad u_n(0, \theta) < \infty. \quad (4)$$

Separating variables in equation (3), we arrive at the singular spectral problem for the equation

$$(xy')' + (\lambda/x - x)y = 0, \quad 0 < x < \alpha, \quad (5)$$

with boundary conditions

$$y(\alpha) = 0, \quad y(0) < \infty, \quad \alpha = \frac{n\pi a}{l}. \quad (6)$$

To investigate the nature of the spectrum, suppose first that the eigenvalues λ are complex. The corresponding eigenfunctions are expressed in terms of Bessel functions of the first kind of imaginary argument,

$$y(x) = I_{\sqrt{-\lambda}}(x), \quad \operatorname{Re} \sqrt{-\lambda} > 0, \quad (7)$$

where the values of λ are the roots of the equation

$$I_{\sqrt{-\lambda}}(\alpha) = 0. \quad (8)$$

Then, putting in the integral relation

$$\int_0^\alpha I_{\sqrt{-\lambda_1}}(x) I_{\sqrt{-\lambda_2}}(x) \frac{dx}{x} = 0 \quad (\lambda_1 \neq \lambda_2) \quad (9)$$

$\lambda_{1,2} = \beta \pm i\gamma$, we arrive at a contradiction, since the left-hand side of (9) is essentially positive. Suppose next that λ is a negative real number. In this case the eigenfunctions are still given by formulas (7), with $\sqrt{-\lambda} > 0$, and (8); however, the latter equality is impossible, since $I_{\sqrt{-\lambda}}(\alpha) > 0$. Finally, for real positive values $\lambda = \tau^2$ there exist real eigenfunctions

$$y(x, \tau) = K_{i\tau}(\alpha) I_{i\tau}(x) - I_{i\tau}(\alpha) K_{i\tau}(x), \quad \tau > 0 \quad (10)$$

($K_\nu(x)$ is the Macdonald function), satisfying both equation (5) and the conditions (6).

Thus, the problem under consideration has a continuous spectrum of eigenvalues, on the basis of which the function u_n is represented in the form

$$u_n = \int_0^\infty [F_1(\tau) \operatorname{sh}(\theta_2 - \theta)\tau + F_2(\tau) \operatorname{sh}(\theta - \theta_1)\tau] y(x, \tau) \frac{d\tau}{\operatorname{sh}(\theta_2 - \theta_1)\tau}. \quad (11)$$

It remains to satisfy the inhomogeneous boundary conditions, which leads to the necessity of determining the quantities $F(\tau)$ from two relations of the form

$$f(x) = \int_0^\infty F(\tau) y(x, \tau) d\tau \quad (0 < x < \alpha), \quad (12)$$

where the functions $f(x)$ are expressed through the prescribed functions $\Phi(r, z)$ as follows*

$$f(x) = \frac{2}{l} \int_0^l \Phi\left(\frac{lx}{n\pi}, z\right) \sin n\pi \frac{z}{l} dz. \quad (13)$$

2°. The formula that allows one to find $F(\tau)$ from the expansion (12) has the form

$$F(\tau) = \frac{2\tau \operatorname{sh} \pi\tau}{\pi^2 |I_{i\tau}(\alpha)|^2} \int_0^\alpha f(x) y(x, \tau) \frac{dx}{x} \quad (14)$$

and, apparently, is new. It can be obtained comparatively simply by a method similar to that indicated in the monograph (1).

For this purpose let us consider the partial differential equation for a certain function $w(x, t)$

$$x \frac{\partial}{\partial x} \left(x \frac{\partial w}{\partial x} \right) - x^2 w - \frac{\partial w}{\partial t} = 0, \quad 0 < x < \alpha, \quad t > 0, \quad (15)$$

and give its solution satisfying the conditions

$$w(\alpha, t) = 0, \quad w(0, t) < \infty, \quad w(x, 0) = f(x). \quad (16)$$

Applying the Laplace transform

$$\bar{w}(x, p) = \int_0^\infty w(x, t) e^{-pt} dt, \quad (17)$$

we find

$$w = \int_0^\alpha f(\xi) \frac{d\xi}{\xi} \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} G(x, \xi, p) e^{pt} \frac{dp}{I_{\sqrt{p}}(\alpha)} \quad (\operatorname{Re} \sqrt{p} > 0), \quad (18)$$

* The method presented is also applicable to the case of a wedge unbounded in the direction of the Oz axis; in this case the series in formula (2) is replaced by an integral, and the right-hand side of (13) by the Fourier transform.

where

$$G(x, \xi, p) = \begin{cases} I_{\sqrt{p}}(x) [I_{\sqrt{p}}(\alpha) K_{\sqrt{p}}(\xi) - I_{\sqrt{p}}(\xi) K_{\sqrt{p}}(\alpha)], & 0 < x \leq \xi, \\ I_{\sqrt{p}}(\xi) [I_{\sqrt{p}}(\alpha) K_{\sqrt{p}}(x) - I_{\sqrt{p}}(x) K_{\sqrt{p}}(\alpha)], & \xi \leq x < \alpha. \end{cases} \quad (19)$$

The complex integral entering into (18) is transformed into real integrals along the banks of the cut made along the negative part of the real axis of the complex plane, where $p = -\tau^2$. Taking into account the fact proved above that the equation (8) has no roots, and also the relation

$$K_{i\tau}(x) = \frac{\pi}{2i} \frac{I_{-i\tau}(x) - I_{i\tau}(x)}{\operatorname{sh} \pi\tau}, \quad (20)$$

after some calculations we obtain

$$w = \frac{2}{\pi^2} \int_0^\alpha f(\xi) \frac{d\xi}{\xi} \int_0^\infty e^{-\tau^2 t} y(x, \tau) y(\xi, \tau) \frac{\tau \operatorname{sh} \pi\tau}{|I_{i\tau}(\alpha)|^2} d\tau. \quad (21)$$

Changing in the last formula the order of integration and then putting $t = 0$, we obtain the expansion

$$f(x) = \frac{2}{\pi^2} \int_0^\infty y(x, \tau) \frac{\tau \operatorname{sh} \pi\tau}{|I_{i\tau}(\alpha)|^2} d\tau \int_0^\alpha f(\xi) y(\xi, \tau) \frac{d\xi}{\xi}, \quad (22)$$

comparison of which with (12) gives the required formula (14).*

3°. If $F(\tau)$ is regarded as the integral transform of $f(x)$ with respect to the eigenfunctions $y(x, \tau)$, then (12) is the inversion formula for this transform.

The pair of reciprocal formulas (12) and (14) gives a generalization of the Kontorovich-Lebedev integral transform (2) to the case of a finite interval. As $\alpha \rightarrow \infty$, the expansion (22) passes into the formula

$$f(x) = \frac{2}{\pi^2} \int_0^\infty K_{i\tau}(x) \tau \operatorname{sh} \pi\tau d\tau \int_0^\infty f(\xi) K_{i\tau}(\xi) \frac{d\xi}{\xi}, \quad 0 < x < \infty,$$

given in the works of N. N. Lebedev (3,4).

Expansions analogous to (22) will also hold in those cases when homogeneous boundary conditions of the second or third kind are prescribed at $x = \alpha$.

Let us note in conclusion that the integral transforms considered in this paper, along with the Dirichlet problem, can also be applied to the solution of mixed problems for equations of the form

$$\Delta u = A \frac{\partial^2 u}{\partial t^2} + B \frac{\partial u}{\partial t} + Cu + F(r, \theta, z, t)$$

with nonhomogeneous boundary and initial conditions.

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CITED LITERATURE

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- ² M. I. Kontorovich, N. N. Lebedev, ZhETF, 8, 10, 1192 (1938).
- ³ N. N. Lebedev, DAN, 52, No. 8, 661 (1946).
- ⁴ N. N. Lebedev, Prikl. matem. i mekh., 13, 5, 465 (1949).

* The derivation given here is of a formal character; in this connection it is necessary to establish the class of functions for which the expansion found is valid—this is the subject of an independent investigation, which, however, lies beyond the scope of the present paper.

Note: Figure translations are in progress. See original paper for figures.

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